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ON VIBRATIONS OF A BODY WITH MANY CONCENTRATED MASSES NEAR THE BOUNDARY

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We consider the asymptotic behavior of the vibration of a body occupying a region $\Omega \subset \mathbb{R}^3$. The density, which depends on a small parameter ε, is of order $O(1)$ out of certain regions where it is $O(\varepsilon^{-m})$ with $m > 2$. These regions, the concentrated masses with diameter $O(\varepsilon)$, are located near the boundary, at mutual distances $O(\eta)$, with $\eta = \eta(\varepsilon) \to 0$. We impose Dirichlet (respectively Neumann) conditions at the points of $\partial\Omega$ in contact with (respectively, out of) the masses. For the critical size $\varepsilon = O(\eta^2)$, the asymptotic behavior of the eigenvalues of order $O(\varepsilon^{m-2})$ is described via a Steklov problem, where the 'mass' is localized on the boundary, or through the eigenvalues of a local problem obtained from the micro-structure of the problem. We use the techniques of the formal asymptotic analysis in homogenization to determine both problems. We also use techniques of convergence in homogenization, Semigroups theory, Fourier and Laplace transforms and boundary values of analytic functions to prove spectral convergence. In the same framework we study the case $m = 2$ as well as the case when other boundary conditions are imposed on $\partial\Omega$.

1. Introduction

Vibration problems of systems with concentrated masses have recently been approached by several authors: they study the asymptotic behavior of a body containing a small region where the density is very much higher than elsewhere. In Refs. 10 and 18 a problem for the Laplace operator has been studied by using quite different techniques. In Ref. 19 the problem is generalized to the elasticity system. References 4 and 11 studied the problem for a vibrating membrane. A lot of cases appear depending on the dimension of the space and the density of the small region.

In this paper we study the vibrations of a body placed in a domain Ω, which contains many small regions of high density, the so-called 'concentrated masses.' These regions share a part of their boundary with a part Σ of the boundary of Ω. Besides, as the small region size decreases, the number of small regions increases, in an analogous way as it happens in homogenization problems. Therefore, techniques

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of boundary homogenization (cf. Refs. 1, 3, 6, 9 and 17) and spectral perturbation theory (cf. Refs. 15 and 16) are used for the study of this problem. Some results related to Secs. 2–4 and 5.1 have already been announced in Ref. 8.

We shall suppose that the diameter of the small regions B^ε is $O(\varepsilon)$ and the distance between them is $O(\eta)$; ε, η being the parameters such that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ (see Fig. 1). The density is of order $O(\varepsilon^{-m})$ in these regions, B^ε and $O(1)$ elsewhere, $m \geq 2$. We study the limit behavior as $\varepsilon \to 0$, of the eigenvalues of a problem for the Laplace operator with Dirichlet condition on $\Sigma \cap \partial B^\varepsilon$ and Neumann conditions on $\Sigma - \partial B^\varepsilon$. Using the mini-max principle we prove that the eigenvalues are of order $O(\varepsilon^{m-2})$ as stated in Proposition 1 (Sec. 2).

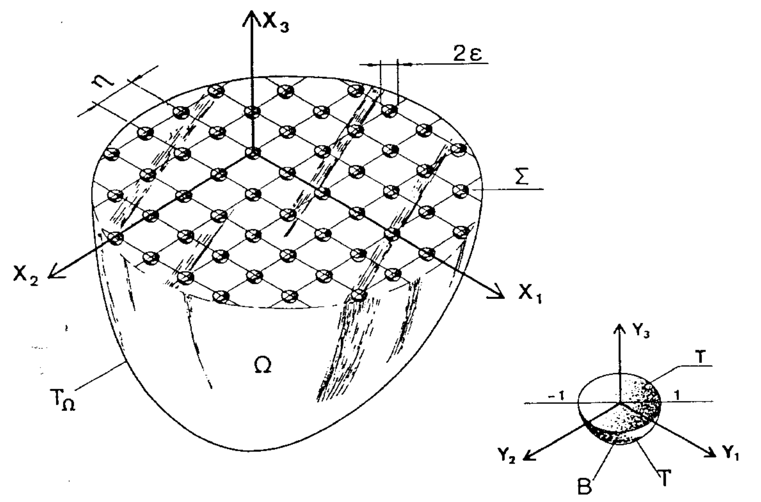

Fig. 1.

Using the method of matched asymptotic expansions we characterize the behavior of the eigenvalues of order $O(\varepsilon^{m-2})$ for the critical size $\varepsilon \cong \alpha\eta^2$, $\alpha > 0$, and $m > 2$: These eigenvalues are of type: $\lambda^\varepsilon = \varepsilon^{m-2}\lambda^0 + \cdots$, where either λ^0 is an eigenvalue of a local problem (problem (3.5)) with an unbounded part without kinetic energy (see Sec. IV.8 of Ref. 15), or $F(\lambda^0)$ is an eigenvalue of a homogenized problem. This homogenized problem is a Steklov type eigenvalue problem (problem (4.5)) where the mass term is localized only on the surface Σ, and the function F is obtained from the micro-structure of the problem in a neighborhood of each concentrated mass (see (3.6)). In the extreme cases, $\varepsilon/\eta^2 \to 0$ and $\varepsilon/\eta^2 \to \infty$, the behavior of these eigenvalues is described only through those of the local problem (see Remarks 3 and 5). We study the local problem and some properties of function F in Sec. 3.

We point out that unlike the case of one concentrated mass, where small eigenvalues (of order $O(\varepsilon^{m-2})$) give rise only to 'local vibrations' (i.e., vibrations only significant in a neighborhood of the concentrated mass), in this case, because of the existence of many concentrated masses, small eigenvalues also give rise to 'global vibrations' (i.e., vibrations affecting all the body).

The results of spectral convergence for $m > 2$ are stated in Theorem 1 (Sec. 5). We justify the formal asymptotic expansions by using the Energy Method (see Sec. 5.1). Besides, we also derive spectral convergence by the knowing properties of time-depending solutions of vibrations problems and by using Fourier and Laplace transforms (see Secs. 5.2 and 5.3).

Quite different limit behavior is obtained for the eigenvalues in the case $m = 2$ (see Sec. 6): the homogenized problem that we obtain is an implicit eigenvalue problem with the spectral parameter appearing both in the equation and in the boundary condition. In the extreme cases eigenvalues of order $O(1)$ can give rise to global and local vibrations (see Sec. 6.1).

In Sec. 7 we study the asymptotic behavior of the eigenvalues when Neumann condition is imposed on all the boundary Σ.

2. Setting of the Problem

Let Ω be a bounded open domain of $\mathbb{R}^3$ situated in the semi-space $\mathbb{R}^{3-} = \{(x_1,x_2,x_3)/x_3 < 0\}$ with a Lipschitz boundary $\partial\Omega = \Sigma \cup \Gamma_\Omega$, its part $\Sigma = \partial\Omega \cap \{x_3 = 0\}$ is assumed to be non-empty.

Let B be the semi-ball $B = \{(y_1,y_2,y_3)/y_1^2 + y_2^2 + y_3^2 < 1,\ y_3 < 0\}$, in the auxiliary space $\mathbb{R}^3$ with coordinates y_1, y_2, y_3. Let ∂B be its boundary $\partial B = T \cup \Gamma$, where T is the circle $T = \{(y_1,y_2,0)/y_1^2 + y_2^2 < 1\}$ in the plane $\{y_3 = 0\}$ (see Fig. 1). Let B^ε ($T^\varepsilon, \Gamma^\varepsilon$ respectively) denote its homothetic $\varepsilon B(\varepsilon T, \varepsilon\Gamma$ respectively) in the x_1, x_2, x_3 space. In order to simplify, if there is no ambiguity, we shall also use B^ε ($T^\varepsilon, \Gamma^\varepsilon$ respectively) to denote the domain obtained by translation of the previous $B^\varepsilon(T^\varepsilon, \Gamma^\varepsilon$ respectively) in the plane $\{x_3 = 0\}$, centered on the points $\tilde{x}_k = (k_1\eta, k_2\eta, 0)$, $k_1, k_2 \in \mathbb{N}$. Both parameters ε, η are positive, with $\varepsilon < \eta$. $\eta = \eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. The number of all the B^ε contained in Ω is $N(\varepsilon)$, with $N(\varepsilon) = O(1/\eta^2)$ (see Fig. 1). The geometric configuration in the plane $\{x_3 = 0\}$ is analogous to that described in Refs. 6, 9 and 17.

We study the asymptotic behavior as $\varepsilon \to 0$, of the eigenvalues of the problem:

$$\begin{cases} -\Delta u^\varepsilon = \rho^\varepsilon(x)\lambda^\varepsilon u^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \Gamma_\Omega \cup \bigcup T^\varepsilon \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \Sigma - \overline{\bigcup T^\varepsilon}, \end{cases} \tag{2.1}$$

where $\rho^\varepsilon = \rho^\varepsilon(x)$ is the function defined as:

$$\rho^\varepsilon(x) = 1/\varepsilon^m \quad \text{if} \quad x \in \bigcup B^\varepsilon \quad \text{and} \quad \rho^\varepsilon(x) = 1 \quad \text{if} \quad x \in \Omega - \overline{\bigcup B^\varepsilon}\,. \tag{2.2}$$

Here and in the sequel the symbol $\bigcup$ is extended, for fixed ε, to all the semi-balls B^ε centered on $\tilde{x}_k$, contained in Ω. The parameter m being $m \geq 2$.

The variational formulation of problem (2.1) is:

Find λ^ε and $u^\varepsilon \in V^\varepsilon$, $u^\varepsilon \neq 0$, satisfying the equation:

$$\int_\Omega \nabla u^\varepsilon \cdot \nabla v^\varepsilon dx = (\lambda^\varepsilon/\varepsilon^m) \int_{\bigcup B^\varepsilon} u^\varepsilon v^\varepsilon dx + \lambda^\varepsilon \int_{\Omega - \overline{\bigcup B^\varepsilon}} u^\varepsilon v^\varepsilon dx \quad \forall\, v^\varepsilon \in V^\varepsilon , \quad (2.3)$$

V^ε being the completion of $\{u \in C^{-1}(\bar{\Omega})/u = 0 \text{ on } \bigcup T^\varepsilon \cup \Gamma_\Omega\}$ in the topology of $H^1(\Omega)$.

Problem (2.3) is a standard eigenvalue problem: Let us consider the sequence of eigenvalues of this problem:

$$0 < \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \cdots \leq \lambda_n^\varepsilon \leq \cdots \underset{n\to\infty}{\longrightarrow} \infty \quad (2.4)$$

(with the classical convention of repeated eigenvalues). Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the corresponding sequence of eigenfunctions, which is assumed to be an orthonormal basis of V^ε.

We find estimates for the eigenvalues of (2.1):

Proposition 1. *For each $i = 1, 2, \ldots, n, \ldots$, we have:*

$$\lambda_i^\varepsilon \leq C_i \varepsilon^{m-2} , \quad (2.5a)$$

$$C\varepsilon^{m-2} \leq \lambda_i^\varepsilon , \quad (2.5b)$$

where C and C_i are constants, C_i independent of ε, and C independent of ε and i.

Proof. (a) Let us prove (2.5a). By the mini-max principle, for each i fixed, we have:

$$\lambda_i^\varepsilon = \min_{E_i^\varepsilon \subset V^\varepsilon} \left\{ \max_{\substack{v \in E_i^\varepsilon \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^2 dx}{\frac{1}{\varepsilon^m}\int_{\bigcup B^\varepsilon} |v|^2 dx + \int_{\Omega - \bigcup B^\varepsilon} |v|^2 dx} \right\} , \quad (2.6)$$

where the minimum is taken over all the subspaces $E_i^\varepsilon \subset V^\varepsilon$, with dim $E_i^\varepsilon = i$.

We take the particular space $E_i^{\varepsilon^*} = [W_1^\varepsilon, W_2^\varepsilon, \ldots, W_i^\varepsilon]$, where $\{W_p^\varepsilon\}_{p=1}^i$ are constructed in the following manner:

Let us consider the eigenvalues $\{\lambda_j\}_{j=1}^\infty$ of the problem:

$$\begin{cases} -\Delta_y W_p = \lambda_p W_p \quad \text{in} \quad B , \qquad (y = x/\varepsilon) \\ \quad W_p = 0 \quad \text{on} \quad \partial B \end{cases}$$

(with the classical convention of repeated eigenvalues). Let $\{W_j\}_{j=1}^\infty$ be the corresponding sequence of eigenfunctions which is assumed to be an orthonormal basis of $L^2(B)$. For each $p = 1, 2, \ldots, i$, we define W_p^ε as

$$W_p^\varepsilon(x) = \begin{cases} W_p\left(\dfrac{x - \tilde{x}}{\varepsilon}\right) \quad \text{if} \quad \left(\dfrac{x - \tilde{x}}{\varepsilon}\right) \in B^\varepsilon \qquad \text{for each } \tilde{x} = \tilde{x}_k \\ 0 \quad \text{if} \quad x \in \Omega - \bigcup B^\varepsilon . \end{cases}$$

It is evident that:

$$\max_{\substack{v \in E_i^{\varepsilon^*} \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^2 dx}{\frac{1}{\varepsilon^m}\int_{\bigcup B^\varepsilon} |v|^2 dx + \int_{\Omega - \bigcup B^\varepsilon} |v|^2 dx} = \max_{\substack{v \in E_i^{\varepsilon^*} \\ v \neq 0}} \frac{\varepsilon^{m-2}\int_B |\nabla v|^2 dy}{\int_B |v|^2 dy} = \lambda_i \varepsilon^{m-2} ,$$

so from (2.6), inequality (2.5a) follows.

(b) In order to prove (2.5b), we consider the Poincaré's inequality for $\{u \in H^1(B)/ u = 0 \text{ on } T\}$ to obtain:

$$(1/\varepsilon^2) \int_{B^\varepsilon} |v^\varepsilon|^2 dx \leq \text{Cte} \int_{B^\varepsilon} |\nabla v^\varepsilon|^2 dx \quad \forall\, v^\varepsilon \in V^\varepsilon \quad (2.7)$$

(Cte being a constant independent of ε). So, by taking $\lambda^\varepsilon = \lambda_i^\varepsilon$ and $v^\varepsilon = u^\varepsilon = u_i^\varepsilon$ the corresponding eigenfunction in (2.3), on account of (2.7) and the Poincaré inequality for $\{u \in H^1(\Omega)/u = 0 \text{ on } \Gamma_\Omega\}$ we obtain inequality (2.5b). □

Remark 1. We have considered B as a semi-ball for simplicity. Nevertheless, all the results of this paper hold if B is assumed to be a bounded open domain of $\mathbb{R}^{3-}$ with a Lipschitz boundary, and $T = \partial B \cap \{y_3 = 0\}$ is an open domain containing the origin.

3. Local Problem

In this section we study the local problem which gives us microscopic information about the eigenfunctions of (2.1). This problem is posed on the semispace $\mathbb{R}^{3-} = \{y_3 < 0\}$ and leads us to the study of an eigenvalue problem. We also study some properties of a function F defined through the solution of the local problem, and which will appear in the study of asymptotic behavior of the eigenvalues of (2.1). The relation of the local problem and function F with problem (2.1) will be justified in Sec. 4.

Let us introduce the function $V^\lambda = V^\lambda(y)$ depending on the parameter λ:

$$V^\lambda(y) = H^\lambda(y) + W(y) , \quad (3.1)$$

where $W = W(y)$ is the solution of the problem:

$$\begin{cases} -\Delta_y W = 0 \quad \text{in} \quad \mathbb{R}^{3-} \\ W = 0 \quad \text{on} \quad T , \quad \dfrac{\partial W}{\partial n} = 0 \quad \text{on} \quad \{y_3 = 0\} - \bar{T} \\ W(y) \longrightarrow 1 , \quad |y| \to \infty , \quad y_3 < 0 , \end{cases} \quad (3.2)$$

considered in Sec. V of Ref. 13 (cf. also Ref. 6), and $H^\lambda(y)$ is the solution of the 'local problem':

$$\begin{cases} -\Delta_y H = \lambda H + \lambda W \quad \text{in} \quad B \\ -\Delta_y H = 0 \quad \text{in} \quad \mathbb{R}^{3-} - \bar{B} \\ [H] = \left[\dfrac{\partial H}{\partial n}\right] = 0 \quad \text{on} \quad \Gamma \\ H = 0 \quad \text{on} \quad T\,, \quad \dfrac{\partial H}{\partial n} = 0 \quad \text{on} \quad \{y_3 = 0\} - \bar{T} \\ H(y) \longrightarrow 0\,, \quad |y| \to \infty\,, \quad y_3 < 0 \end{cases} \tag{3.3}$$

which contains the parameter λ, and the brackets denote the jump across Γ.

Problem (3.3) has an equivalent variational formulation:

Find $H^\lambda \in \mathcal{V}$, satisfying:

$$\int_B \nabla H^\lambda \cdot \nabla V dy + \langle \mathcal{T} H^\lambda|_\Gamma, V|_\Gamma \rangle = \lambda \int_B H^\lambda V dy + \lambda \int_B W V dy \quad \forall\, V \in \mathcal{V}\,, \tag{3.4}$$

where $\mathcal{V}$ is the complete space of $\{u \in C^1(\overline{B})/u = 0 \text{ on } T\}$ with the norm of $H^1(B)$, and operator $\mathcal{T} \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ is the 'normal derivative' operator (see Sec. IV.8 of Ref. 15 for definition and properties of $\mathcal{T}$).

Let us consider the eigenvalue problem associated with (3.4):

Find ν and $U \in \mathcal{V}$, $U \neq 0$, satisfying:

$$\int_B \nabla U \cdot \nabla V dy + \langle \mathcal{T} U|_\Gamma, V|_\Gamma \rangle = \nu \int_B U V dy \quad \forall\, V \in \mathcal{V}\,. \tag{3.5}$$

Problem (3.5) has a countable infinity of positive eigenvalues (see Sec. IV.8 of Ref. 15 for a study of this kind of problems).

Provided λ is not an eigenvalue of (3.5), problem (3.4) has a unique solution in $\mathcal{V}$. In this case we have:

$$H^\lambda = (\mathcal{A} - \lambda)^{-1} \lambda W\,,$$

$\mathcal{A}$ being the operator associated with the form defined on $\mathcal{V}$ on the left-hand side of (3.5). Besides, on account of (3.1), we can also define the function $F(\lambda)$ as:

$$\begin{aligned} F(\lambda) &= -\left\langle \frac{\partial V^\lambda}{\partial n_y}|_\Gamma, 1 \right\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\ &= -\left\langle \frac{\partial}{\partial n_y}\left((\mathcal{A}-\lambda)^{-1}\lambda W + W\right)|_\Gamma, 1 \right\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\,, \end{aligned} \tag{3.6}$$

where $\vec{n}_y$ denotes the unit outer normal vector to Γ.

We give some properties of the function F that we shall use in the following sections.

Proposition 2. *The function $F(\lambda)$ defined in (3.6), for $\lambda \in \mathbb{C}$, is a meromorphic function with positive real poles $\{\nu_i\}_{i=1}^\infty$, which are eigenvalues of (3.5). Moreover, for each $i \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, it satisfies:*

$$\lim_{\lambda \to \nu_i^+} F(\lambda) = +\infty \quad \text{and} \quad \lim_{\lambda \to \nu_i^-} F(\lambda) = -\infty\,.$$

Proof. On account of (3.6), it is evident that the only possible singular points of F are those of the spectrum of $\mathcal{A}$.

Let ν_i be a fixed eigenvalue of (3.5). Considering the Laurent expansion of the resolvent operator $(\mathcal{A}-\lambda)^{-1}$ in a reduced neighborhood of this value, we can write:

$$\frac{\partial}{\partial n_y}(\mathcal{A}-\lambda)^{-1}\lambda W\Big|_\Gamma = \frac{\partial}{\partial n_y}\left(\lambda \frac{\mathcal{P}_i W}{\lambda - \nu_i}\right)\Big|_\Gamma + \text{analytic part in } \lambda\,,$$

where $\mathcal{P}_i$ is a projection of the space $L^2(B)$ on the eigenspace associated with ν_i, i.e.,

$$\frac{\partial}{\partial n_y}(\mathcal{A}-\lambda)^{-1}\lambda W\Big|_\Gamma = \lambda \frac{1}{\lambda-\nu_i}\frac{\partial}{\partial n_y}\left(\sum_j (W, U_j) U_j\right)\Big|_\Gamma + \text{analytic part in } \lambda\,, \tag{3.7}$$

where the summation is extended from $j = 1, 2, \ldots, m_i; m_i$ being the multiplicity of the eigenvalue ν_i, and U_j the corresponding eigenfunctions associated with ν_i, supposed orthonormal in $L^2(B)$, and $(W, U_j) = \int_B W \cdot U_j dy$.

Taking $V = U = U_j$, for each $j = 1, 2, \ldots, m_i$, and $\nu = \nu_i$ in (3.5) we show that (3.7) can be written as

$$\frac{\partial}{\partial n_y}(\mathcal{A}-\lambda)^{-1}\lambda W\Big|_\Gamma = -\lambda \frac{1}{\lambda-\nu_i}\frac{1}{\nu_i}\sum_j \left\langle \frac{\partial U_j}{\partial n_y}\Big|_\Gamma, 1 \right\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \frac{\partial U_j}{\partial n_y}\Big|_\Gamma + \text{analytic part in } \lambda\,.$$

Therefore, from (3.6), in a reduced neighbourhood of ν_i we have:

$$F(\lambda) = \lambda \frac{1}{\lambda-\nu_i}\frac{1}{\nu_i}\sum_j \left(\left\langle \frac{\partial U_j}{\partial n_y}|_\Gamma, 1 \right\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\right)^2 + g(\lambda)\,, \tag{3.8}$$

where $g(\lambda)$ is an analytic function of λ.

We note that if W is orthogonal in $L^2(B)$ to the eigenspace associated with ν_i, then F is an analytic function in a neighborhood of ν_i. Nevertheless, it is easy to prove that there are countable number of eigenvalues of (3.5), such that W is not orthogonal in $L^2(B)$ to the eigenspace associated with them. So, the result of the proposition follows from (3.8). □

Remark 2. By the Fredholm alternative, if $\lambda = \nu$ is an eigenvalue of (3.5), and W is orthogonal to the eigenspace associated with ν, then (3.4) has a solution H, but it is not unique. Nevertheless, as we can see in the proof of proposition 1, $F(\nu)$ exists and it is unique (see Eqs. (3.6) and (3.8)). □

In order to simplify the calculations in Secs. 5 and 6, it will prove useful to write $F(\lambda)$ as:

$$F(\lambda) = -\int_{\mathbb{R}^{3-}} |\nabla U^\lambda|^2 dy + \lambda \int_B (U^\lambda - 1)^2 dy\,, \tag{3.9}$$

where $U^\lambda(y) = 1 - V^\lambda(y)$ is now the solution of the problem:

$$\begin{cases} -\Delta_y U = \lambda U - \lambda & \text{in } B \\ -\Delta_y U = 0 & \text{in } \mathbb{R}^{3-} - \bar{B} \\ [U] = \left[\frac{\partial U}{\partial n}\right] = 0 & \text{on } \Gamma \\ U = 1 \text{ on } T, \quad \frac{\partial U}{\partial n} = 0 & \text{on } \{y_3 = 0\} - \bar{T} \\ U(y) \longrightarrow 0, \quad |y| \to \infty, & y_3 > 0. \end{cases} \tag{3.10}$$

From the variational formulation of (3.10), Eq. (3.9) follows. On the other hand, it is evident that U^λ can be extended to a harmonic function out of unit ball. Therefore, we have the estimates: $U^\lambda(y) = O(|y|^{-1})$, $\frac{\partial U^\lambda}{\partial y}(y) = O(|y|^{-2})$ in a neighborhood of infinity.

In the sequel if there is no ambiguity we shall call the eigenvalues of (3.5), 'eigenvalues of the local problem.'

4. Asymptotic Expansions and Homogenized Problem

In order to have an idea about the asymptotic behavior of the eigenelements of problem (2.1) we shall apply the techniques of asymptotic matched expansions. In this section we shall consider $m > 2$.

On accord of relations (2.5) we postulate an expansion of $\lambda^\varepsilon = \lambda_i^\varepsilon$ for a given $i \in \mathbb{N}$:

$$\lambda^\varepsilon = \varepsilon^{m-2}\lambda^0 + \lambda^1\varepsilon^{m-1} + \cdots, \tag{4.1a}$$

we also postulate for the corresponding eigenfunction u^ε an outer expansion in Ω of the form:

$$u^\varepsilon(x) = u^0(x) + \varepsilon u^1(x) + \cdots, \tag{4.1b}$$

where u^0 satisfies the conditions: it is a harmonic function in $\Omega, u^0 = 0$ on Γ_Ω and some boundary condition on Σ that we will obtain using the matching relations with the local expansions, in an analogous manner as it was performed in Refs. 6 and 17:

Introducing the local variable $y = (x - \tilde{x})/\varepsilon$ in a neighborhood of $\tilde{x} = \tilde{x}_k$ (for each k fixed), we postulate a local expansion for $u^\varepsilon, u^\varepsilon = v^0(y) + \varepsilon v^1(y) + \cdots$. The matching condition with outer expansion (4.1b) allows us to obtain microscopic information of solution u^ε in a neighborhood of $B^\varepsilon : v^0(y) = u^0(\tilde{x})V^{\lambda^0}(y)$, where V^{λ^0} is the function defined in (3.1) for $\lambda = \lambda^0$.

Using this microscopic information of u^ε, we consider a local asymptotic expansion, for the derivatives of u^ε, in a neighborhood of Σ. The matching condition with the outer expansion gives us:

$$\left.\frac{\partial u^0}{\partial x_3}\right|_{x_3=0^-} = -\lim_{\varepsilon\to 0}\sum_{\tilde{x}}\frac{1}{\varepsilon}u^0(\tilde{x})T_x\left.\left(\frac{\partial V^{\lambda^0}}{\partial n_y}\right)\right|_{\Gamma_\varepsilon}, \tag{4.2}$$

where T_x denotes the change of variables from y to x, and the summation is extended to all the centers $\tilde{x}$ of B^ε contained in Ω (see Refs. 6 and 17 for more details about this kind of techniques).

By calculating the limit on the right-hand side of (3.4) we obtain:

$$\left.\frac{\partial u^0}{\partial x_3}\right|_\Sigma = -\alpha\left\langle \left.\frac{\partial V^{\lambda^0}}{\partial n_y}\right|_\Gamma, 1\right\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} u^0|_\Sigma, \tag{4.3}$$

where $\alpha = \lim_{\varepsilon\to 0}(\varepsilon/\eta^2) > 0$.

On account of definition (3.6), the condition verified by u^0 on Σ is

$$\left.\frac{\partial u^0}{\partial x_3}\right|_\Sigma = \alpha F(\lambda^0)u^0|_\Sigma. \tag{4.4}$$

So, the calculations performed in this section allow us to affirm that, if λ^0 is not an eigenvalue of the local problem (3.5) (see also Sec. 3), then $(F(\lambda^0), u^0)$ must be an eigenelement of the Steklov type problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_\Omega \\ \frac{\partial u}{\partial n} = \mu\alpha u & \text{on } \Sigma. \end{cases} \tag{4.5}$$

This problem has an equivalent variational formulation:
Find μ and $u \in \mathrm{V}, u \neq 0$, satisfying:

$$\int_\Omega \nabla u \cdot \nabla v dx = \alpha\mu\int_\Sigma uv d\Sigma \quad \forall\, v \in \mathrm{V}. \tag{4.6}$$

V being the completion of $\{u \in C^1(\bar{\Omega})/u = 0 \text{ on } \Gamma_\Omega\}$ in the topology of $H^1(\Omega)$.

Problem (4.6) can be considered as an eigenvalue problem for a compact and self-adjoint operator A defined on V. So, there is a countable number of positive eigenvalues, $\{\mu_i\}_{i=1}^\infty$, of (4.6) converging to infinity, as $i \to \infty (1/\mu_i$ eigenvalue of $A)$. This result leads us to consider the 'characteristic equation':

$$F(\lambda) = \mu_k \tag{4.7}$$

for each $k \in \mathbb{N}$ (see Ref. 12 for other characteristic equations).

As an immediate consequence of Proposition 2 we have:

Proposition 3. *For each μ_k eigenvalue of (4.6), there is a positive root of (4.7).*

This result ensures the consistency of the preceding formal expansions where, for the critical size $\varepsilon \simeq \alpha\eta^2$, we must take λ^0 (in (4.1a)) equal to an eigenvalue of (3.5) or a solution of (4.7) for each eigenvalue μ_k of (4.5).

We point out that in (4.5) the 'term mass' (i.e., the coefficient of the spectral parameter μ) appears only on the surface Σ. This term is obtained from a homogenization of the point masses. By analogy with homogenization problems we shall call the problem (4.5) the 'homogenized problem.'

Remark 3. In the extreme case $\varepsilon/\eta^2 \to 0$ ($\varepsilon/\eta^2 \to \infty$ respectively) the condition satisfied by u^0 on Σ is $u^0 = 0$($\frac{\partial u^0}{\partial n} = 0$ on Σ respectively), i.e., $u^0 \equiv 0$. It seems that, in these cases, the eigenvalues of order $O(\varepsilon^{m-2})$ of problem (2.1) give rise to local vibrations of each mass independent of the others, and not to global vibrations of the body.

5. Results of Convergence

In this section we shall use the Energy Method, Semigroups theory, Fourier series, Fourier and Laplace transforms and results about boundary values of analytic functions, in order to give results of convergence of eigenvalues of order (ε^{m-2}) of (2.1). Throughout this section we shall consider $m > 2$ and $\lim_{\varepsilon\to 0}(\varepsilon/\eta^2) = \alpha > 0$.

We show, that some of these eigenvalues can be approached through eigenvalues of the local problem (3.5), and others through roots of the characteristic equation (4.7). We state the main results of this section in the following theorem:

Theorem 1. *Let $\lambda^{\varepsilon_n}_{i(\varepsilon_n)}$ be a sequence of eigenvalues of (2.1) such that $\lambda^{\varepsilon_n}_{i(\varepsilon_n)}/\varepsilon_n^{m-2}$ converges to λ^0 as $\varepsilon_n \to 0$. Then, λ^0 is an eigenvalue of (3.5) or $F(\lambda^{0^n})$ is an eigenvalue of (4.5). Reciprocally, if λ^0 is an eigenvalue of (3.5), or λ^0 is a root of equation (4.7) such that $F'(\lambda^0) \neq 0$, then λ^0 is a point of accumulation of $\lambda_i^\varepsilon/\varepsilon^{m-2}$ (λ_i^ε being eigenvalues of (2.1)).*

The proof of this theorem is a direct consequence of Theorems 2, 3 and 4 that we shall prove in Secs. 5.1, 5.2 and 5.3 respectively.

5.1. *Convergence to the spectrum of the homogenized problem*

This section is devoted to proving the first result of Theorem 1. This result justifies in some way the formal calculations performed in Sec. 4, and it was already announced in Ref. 8 without proof. We use the Energy Method to prove it (see Refs. 7 and 9 in relation to its usage in homogenization problems).

Theorem 2. *Let λ_i^ε and u_i^ε be the eigenelements of problem (2.1). Let us suppose that there is a sequence $\varepsilon_n \to 0$ such that:*

$$(\lambda_i^{\varepsilon_n}/\varepsilon_n^{m-2}) \underset{n\to\infty}{\longrightarrow} \lambda^0$$
$$u_i^{\varepsilon_n} \underset{n\to\infty}{\longrightarrow} u^0\,, \quad \text{weakly in } H^1(\Omega)\,,$$

where the index i may depend on ε_n, and λ^0 is assumed not to be an eigenvalue of the local problem (3.5). Then $\mu = F(\lambda^0)$ and u^0 are eigenelements of the homogenized problem (4.5).

Proof. Throughout this proof we shall write, for simplicity, $(\lambda^\varepsilon, u^\varepsilon)$ instead of $(\lambda_i^{\varepsilon_n}, u_i^{\varepsilon_n})$. We shall do this proof in two steps; in the first one we give some results which will simplify the proof; in the second we show the result of the proposition.

Step 1. As in homogenization problem, taking into account the structure of the solution of the local problem (3.10), we construct a sequence of 'test functions' w^ε which allows us to take limits in (2.3). We use the same kind of techniques as in Refs. 3 and 13.

Let $U = U^{\lambda_0}$ be the solution of problem (3.10) for $\lambda = \lambda^0$, we shall denote by U^ε the function $U^\varepsilon(x) = U(\frac{x}{\varepsilon})$. Let φ^ε be the smooth function which takes the value 1 in the semi-ball of radius $(\varepsilon + \eta/8), B(\varepsilon + \eta/8)$, and is zero out of the semi-ball of radius $(\varepsilon + \eta/4), B(\varepsilon + \eta/4)$:

$$\varphi^\varepsilon(x) = \varphi\left(2\frac{|x| - \varepsilon}{\eta}\right)\,,$$

where $\varphi \in C^\infty[0,1], 0 \le \varphi \le 1, \varphi = 1$ in $[0,1/4]$ and $\mathrm{Supp}(\varphi) \subset [0,1/2]$.

Let us consider the function $w^\varepsilon = (1 - U^\varepsilon\varphi^\varepsilon)$ that we prolong by periodicity over all the semi-balls $B(\varepsilon + \eta/4)$ centered on $\tilde{x}$ contained in Ω, and by value 1 outside. We have:

(a) $w^\varepsilon \in H^1(\Omega)$, $w^\varepsilon|_{\cup B^\varepsilon \cup \Gamma_\Omega} = 0$ and $w^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 1$ weakly in $H^1(\Omega)$.

(b) For each $v \in V$ and each sequence $\{v^\varepsilon\}_\varepsilon$ with $v^\varepsilon \in V^\varepsilon$ and $v^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} v$ weakly in $H^1(\Omega)$, there exists a sequence $\tilde{v}^\varepsilon \in H^1(\Omega), \tilde{v}^\varepsilon|_{\cup B^\varepsilon \cup \Gamma_\Omega} = 0$, such that $\tilde{v}^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} v$ weakly in $H^1(\Omega)$ and satisfies the relations:

$$\lim_{\varepsilon\to 0}\int_\Omega \nabla\tilde{v}^\varepsilon \cdot \nabla w^\varepsilon \phi dx = \alpha\int_{\mathbb{R}^{3-}} |\nabla U|^2 dy \int_\Sigma v\phi d\Sigma\,, \tag{5.1}$$

$$\lim_{\varepsilon\to 0}(\lambda^\varepsilon/\varepsilon^m)\int_{\cup B^\varepsilon} \tilde{v}^\varepsilon w^\varepsilon \phi dx = \alpha\lambda^0\int_B (1-U)^2 dy \int_\Sigma v\phi d\Sigma\,, \tag{5.2}$$

$$\lim_{\varepsilon\to 0}\left[(\lambda^\varepsilon/\varepsilon^m)\int_{\cup B^\varepsilon}(v^\varepsilon - \tilde{v}^\varepsilon)w^\varepsilon\phi dx - \int_{\cup B(\varepsilon+\eta/4)}\nabla(v^\varepsilon - \tilde{v}^\varepsilon)\nabla w^\varepsilon \phi dx\right] = 0\,. \tag{5.3}$$

ϕ being any function of $\{u \in C^1(\bar{\Omega})/u = 0 \text{ on } \Gamma_\Omega\}$.

Property (a) is shown as in Refs. 3 and 13 (Sec. 3). In order to prove property (b) we use the usual techniques in boundary homogenization problems (cf. Refs. 1, 3 and 13):

For each fixed $h > 0$, let us consider a regular triangulation of the domain Ω composed of tetrahedrons of diameter h (see Ref. 14 for example). Let $\Pi_h u$ denote the projection of the element $u \in H^1(\Omega)$ on the space of the continuous functions over $\bar{\Omega}$ which are polynomial of degree 1 on each tetrahedron and take value zero on Γ_Ω. Let $v^{\varepsilon,h}$ be the function $v^{\varepsilon,h} = (\Pi_h v^\varepsilon)w^\varepsilon$. By a process of taking limits, first as $\varepsilon \to 0$, and later as $h \to 0$, we may choose a sequence $h(\varepsilon)$, with $h(\varepsilon) \to 0$

as $\varepsilon \to 0$, and with the functions $\tilde{v}^\varepsilon = (\Pi_{h(\varepsilon)}v^\varepsilon)w^\varepsilon$ converging to v weakly in $H^1(\Omega)$ and satisfying Eqs. (5.1) and (5.2) (see p. 37 of Ref. 1 and Refs. 3 and 13, Sec. 3 for details of the proof). Relation (5.3) is proved on the basis of the variational formulation of (3.10) and the estimates for $U^\varepsilon, \varphi^\varepsilon$ and their derivatives on $B(\varepsilon+\eta/4) - B(\varepsilon+\eta/8)$.

Step 2. Let us consider $\phi \in \{u \in C^1(\bar{\Omega})/u = 0 \text{ on } \Gamma_\Omega\}$. By taking $v^\varepsilon = \phi w^\varepsilon$ in (2.3), because of $\lambda^\varepsilon = O(\varepsilon^{m-2})$, we have:

$$\int_\Omega \nabla u^\varepsilon \cdot \nabla w^\varepsilon \phi dx + \int_\Omega \nabla u^\varepsilon \cdot \nabla \phi w^\varepsilon dx = (\lambda^\varepsilon/\varepsilon^m)\int_{\cup B^\varepsilon} u^\varepsilon w^\varepsilon \phi dx + o(1)\,. \quad (5.4)$$

Let $\tilde{u}^\varepsilon$ be the sequence constructed in Step 1, (b). Considering (5.1) and the convergence of sequences $u^\varepsilon, w^\varepsilon$ weakly in $H^1(\Omega)$ as $\varepsilon \to 0$, we obtain in (5.4):

$$\begin{aligned}\int_\Omega \nabla u^0 \nabla \phi dx &+ \alpha \int_{\mathbb{R}^{3-}} |\nabla U|^2 dy \int_\Sigma u^0 \phi d\Sigma \\ &= \lim_{\varepsilon=0}\left[(\lambda^\varepsilon/\varepsilon^m)\int_{\cup B^\varepsilon} u^\varepsilon w^\varepsilon \phi dx - \int_\Omega \nabla(u^\varepsilon - \tilde{u}^\varepsilon)\cdot \nabla w^\varepsilon \phi dx\right]\,. \end{aligned} \quad (5.5)$$

For each $\psi \in \{u \in C^1(\bar{\Omega})/u = 0 \text{ on } \Gamma_\Omega\}$ we consider the function ψw^ε and the relation:

$$\int_{\cup B^\varepsilon} u^\varepsilon w^\varepsilon \phi dx = \int_{\cup B^\varepsilon} \phi\psi w^{\varepsilon 2} dx + \int_{\cup B^\varepsilon} (u^\varepsilon - \psi w^\varepsilon) w^\varepsilon \phi dx\,.$$

Taking into account that $\varphi^\varepsilon = 1$ in B^ε, $\varepsilon \simeq \alpha\eta^2$, ϕ and ψ are smooth functions, $(\lambda^\varepsilon/\varepsilon^{m-2}) \underset{\varepsilon\to 0}{\longrightarrow} \lambda^0$, and (5.2) we have:

$$\begin{aligned}\lim_{\varepsilon\to 0}(\lambda^\varepsilon/\varepsilon^m)\int_{\cup B^\varepsilon} u^\varepsilon w^\varepsilon \phi dx &= \alpha\lambda^0 \int_B (1-U)^2 dy \int_\Sigma \phi\psi d\Sigma \\ &+ \lim_{\varepsilon\to 0}(\lambda^\varepsilon/\varepsilon^m)\int_{\cup B^\varepsilon}(u^\varepsilon - \tilde{u}^\varepsilon)w^\varepsilon\phi dx + \alpha\lambda^0\int_B(1-U)^2 dy\int_\Sigma (u^0-\psi)\phi d\Sigma\,. \end{aligned} \quad (5.6)$$

In relation (5.6) we make ψ tend to u^0 in $H^1(\Omega)$ and we substitute it in (5.5). On account of (5.3) ($\varphi^\varepsilon = 0$ out of the semi-balls $B(\varepsilon+\eta/4)$), and relation (3.9) we deduce:

$$\int_\Omega \nabla u^0 \nabla \phi dx = \alpha F(\lambda^0)\int_\Sigma u^0 \phi d\Sigma$$

and the result of the theorem is proved. □

Remark 4. If, in Theorem 2, λ^0 is an eigenvalue of (3.5) and W is orthogonal to the eigenspace associated with λ^0, then the result of theorem also holds (cf. Remark 3). □

We give complementary results of Theorem 2 in Secs. 5.2 and 5.3.

5.2. *Local vibrations*

According to Sec. V.12 of Ref. 15 in this section we shall use the idea about obtaining spectral properties of problems depending on a parameter when properties of time-depending solutions of vibrations problems are known. We shall introduce two hyperbolic problems associated with (2.3) and (3.5) respectively and we shall prove, by using Fourier transform, that each eigenvalue of the local problem (3.5) can be approached by sequences $\lambda^\varepsilon_{i(\varepsilon)}/\varepsilon^{m-2}$ as $\varepsilon \to 0$, λ_i^ε being eigenvalues of (2.3). We point out that special initial conditions must be chosen for these problems, in order to prove convergence of the time-depending solutions.

Let us change the variable in (2.3) by setting $y = x/\varepsilon$. We obtain:

$$\int_{\Omega_\varepsilon} \nabla_y U^\varepsilon \cdot \nabla_y V^\varepsilon dy = \gamma_i^\varepsilon \int_{\Omega_\varepsilon} \beta^\varepsilon(y) U^\varepsilon V^\varepsilon dy \quad \forall\, V^\varepsilon \in \tilde{\mathbb{V}}^\varepsilon\,, \quad (5.7)$$

Ω_ε being the space $\{y/\varepsilon y \in \Omega\}$, $\gamma_i^\varepsilon = \lambda_i^\varepsilon/\varepsilon^{m-2}$, and $\beta^\varepsilon(y)$ the function defined as:

$$\beta^\varepsilon(y) = 1 \quad \text{if} \quad y \in \bigcup T_y B^\varepsilon \quad \text{and} \quad \beta^\varepsilon(y) = \varepsilon^m \quad \text{if} \quad y \in \Omega_\varepsilon - \overline{\bigcup T_y B^\varepsilon}\,,$$

where $T_y B^\varepsilon$ denote the transformed domains of semi-balls B^ε contained in Ω to the variable y. $\tilde{\mathbb{V}}^\varepsilon$ is the functional space: $\{U = U(y)/U(\varepsilon y) \in \mathbb{V}^\varepsilon\}$. Let $\{\gamma_i^\varepsilon, E_i^\varepsilon\}_{i=1}^\infty$ be the eigenelements of (5.7), where $\{E_i^\varepsilon\}_{i=1}^\infty$ is assumed to be an orthonormal basis of $\tilde{\mathbb{V}}^\varepsilon$.

It proves to be useful, in this section, to write the problem (3.5) in the form: Find $\nu > 0$ and $U \in \tilde{\mathbb{V}}, U \neq 0$, satisfying:

$$\int_{\mathbb{R}^{3-}} \nabla_y U \cdot \nabla_y V dy = \nu \int_B UV dy \quad \forall\, V \in \tilde{\mathbb{V}}\,, \quad (5.8)$$

where $\tilde{\mathbb{V}}$ is the complete space of $\{U \in D(\overline{\mathbb{R}}^{3-})/U = 0 \text{ on } T\}$ with the norm $\|\nabla_y U\|_{L^2(\mathbb{R}^{3-})}$. It is evident that $\tilde{\mathbb{V}} \subset L^2(B)$ with dense and compact imbedding. Let $\{\nu_j, U^j\}_{j=1}^\infty$ be the eigenelements of (5.8), where $\{U^j\}_{j=1}^\infty$ is assumed to be an orthonormal basis of $\tilde{\mathbb{V}}$.

For each k, we consider (ν_k, U^k) a fixed eigenelement of (5.8). Let $\hat{\varphi}^\varepsilon(y)$ be the smooth function which takes the value 1 in $B(0, \frac{\varepsilon+\eta/8}{\varepsilon})$ and zero out of $B(0, \frac{\varepsilon+\eta/4}{\varepsilon})$ (see Step 1 of Theorem 2). Let A^ε (A respectively) denote the operator associated with the form defined on $\tilde{\mathbb{V}}^\varepsilon$($\tilde{\mathbb{V}}$ respectively) by the left-hand side of (5.7) ((5.8) respectively). As $U^k\hat{\varphi}^\varepsilon \in \tilde{\mathbb{V}}^\varepsilon$, let us consider the hyperbolic problem associated with (5.7):

$$\begin{cases} \beta^\varepsilon(y)\dfrac{d^2\mathbf{U}^\varepsilon}{dt^2} + A^\varepsilon \mathbf{U}^\varepsilon = 0\,,\ t > 0 \\ \mathbf{U}^\varepsilon(0) = U^k\hat{\varphi}^\varepsilon \\ \dfrac{d\mathbf{U}^\varepsilon}{dt}(0) = 0 \end{cases} \quad (5.9)$$

and the hyperbolic problem associated with (5.8)

$$\begin{cases} \beta(y)\dfrac{d^2\mathbf{U}}{dt^2} + A\mathbf{U} = 0\,,\ t > 0 \\ \mathbf{U}(0) = U^k \\ \dfrac{d\mathbf{U}}{dt}(0) = 0\,, \end{cases} \tag{5.10}$$

where $\beta(y)$ is the function defined as 1 in B and 0 in $\mathbb{R}^{3-} - \bar{B}$.

It is known (see for example Vol. I of Ref. 5) that the solution of (5.9) is the unique function $\mathbf{U}^\varepsilon$ such that $\mathbf{U}^\varepsilon \in L^\infty(0,\infty,\tilde{V}^\varepsilon)$, $\mathbf{U}^{\varepsilon'} \in L^\infty(0,\infty,L^2(\Omega_\varepsilon))$, that satisfies:

$$\mathbf{U}^\varepsilon(0) = U^k \hat{\varphi}^\varepsilon \tag{5.11a}$$

and, for any T fixed (positive or negative):

$$\begin{cases} \displaystyle\int_0^T \left[(\nabla_y \mathbf{U}^\varepsilon, \nabla_y V)_{L^2(\Omega_\varepsilon)}\phi(t) - (\beta^\varepsilon \mathbf{U}^{\varepsilon'}, V)_{L^2(\Omega_\varepsilon)}\phi'(t)\right] dt = 0\,, \\ \forall\, V \in \tilde{V}^\varepsilon \quad \text{and} \quad \forall\, \phi \in \{\psi \in C^1[0,T]/\psi(T) = 0\}\,. \end{cases} \tag{5.11b}$$

In the same way, problem (5.10) has a unique solution $\mathbf{U}$ such that $\mathbf{U} \in L^\infty(0,\infty,\tilde{V})$, $\mathbf{U}' \in L^\infty(0,\infty,L^2(B))$, satisfies:

$$\mathbf{U}(0) = U^k \tag{5.12a}$$

and, for any T fixed (positive or negative):

$$\begin{cases} \displaystyle\int_0^T [(\nabla_y \mathbf{U}, \nabla_y V)_{L^2(\mathbb{R}^{3-})}\phi(t) - (\mathbf{U}', V)_{L^2(B)}\phi'(t)]\, dt = 0\,, \\ \forall\, V \in \tilde{V} \quad \text{and} \quad \forall\, \phi \in \{\psi \in C^1[0,T]/\psi(T) = 0\}\,. \end{cases} \tag{5.12b}$$

We state now the relation between the solution of the problems (5.9) and (5.10), but before, let us observe that the elements of $\tilde{V}^\varepsilon$, prolonged by zero in $\mathbb{R}^{3-} - \Omega_\varepsilon$, are elements of $\tilde{V}$.

Proposition 4. *Let* $\mathbf{U}^\varepsilon$ ($\mathbf{U}$ *respectively*) *be the solution of (5.9) ((5.10) respectively). Then,*

$$\mathbf{U}^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \mathbf{U} \quad \text{in } L^\infty(0,\infty,\tilde{V}) \quad \text{weak-}^* \quad \text{and,}$$
$$\mathbf{U}^{\varepsilon'} \underset{\varepsilon\to 0}{\longrightarrow} \mathbf{U}' \quad \text{in } L^\infty(0,\infty,L^2(B)) \quad \text{weak-}^*.$$

Proof. From the conservation of energy, for each $t \in \mathbb{R}$, we have:

$$\|(\beta^\varepsilon)^{1/2}\mathbf{U}^{\varepsilon'}(t)\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla_y \mathbf{U}^\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} = \|\nabla_y(U^k\hat{\varphi}^\varepsilon)\|_{L^2(\Omega_\varepsilon)}\,. \tag{5.13}$$

On account of estimates for $\hat{\varphi}^\varepsilon(y), U^k(y)$ as $|y| \to \infty$ (see Sec. 3), and their derivatives, and because of $\varepsilon = o(\eta)$, we deduce from (5.13) the relations:

$$\|\mathbf{U}^\varepsilon\|_{L^\infty(0,\infty,\tilde{V}^\varepsilon)} \le C \quad \text{and} \quad \|(\beta^\varepsilon)^{1/2}\mathbf{U}^{\varepsilon'}\|_{L^\infty(0,\infty,L^2(\Omega_\varepsilon))} \le C\,, \tag{5.14}$$

where C is a constant independent of ε.

Therefore, there exists a subsequence of $\mathbf{U}^\varepsilon$ ($\mathbf{U}^{\varepsilon'}$ respectively) which converges to a function $\mathbf{U}^*$ in $L^\infty(0,\infty,\tilde{V})$ weak-* (to $\mathbf{U}^{*'}$ in $L^\infty(0,\infty,L^2(B))$ weak-*, respectively) as $\varepsilon \to 0$. In order to identify $\mathbf{U}^*$ with the solution of (5.10), we take limits in (5.11b) for each $V \in \{V \in D(\overline{\mathbb{R}^{3-}})/V = 0 \text{ on } T\}$. On account of (5.14) and the fact that $\text{Supp}(V) \subset B(0, \frac{\varepsilon+\eta/4}{\varepsilon})$ for ε small, we obtain that $\mathbf{U}^*$ satisfies Eq. (5.12b). Besides, as $U^k\hat{\varphi}^\varepsilon$ converges to U^k in $\tilde{V}$ when $\varepsilon \to 0, \mathbf{U}^*$ also satisfies Eq. (5.12a). Thus, the convergence results of the theorem follow. □

We prove the spectral convergence by using the same kind of techniques as in Sec. XII.3 of Ref. 16 and Sec. VII.12 of Ref. 15.

Theorem 3. *Let ν_k be a fixed eigenvalue of the local problem (3.5). Then ν_k is a point of accumulation of $\lambda_i^\varepsilon/\varepsilon^{m-2}$, λ_i^ε being eigenvalues of (2.1).*

Proof. Let us consider the Fourier series expansion of $\mathbf{U}^\varepsilon(t)$ in $\tilde{V}^\varepsilon(\mathbf{U}(t)$ in $\tilde{V}$, respectively):

$$\mathbf{U}^\varepsilon(t) = \sum_{i=1}^\infty \langle U^k\hat{\varphi}^\varepsilon, E_i^\varepsilon\rangle_{\tilde{V}^\varepsilon} \cos(\sqrt{\gamma_i^\varepsilon}t)E_i^\varepsilon$$

$$\left(\mathbf{U}(t) = \sum_{j=1}^\infty \langle U^k, U^j\rangle_{\tilde{V}} \cos(\sqrt{\nu_j}t)U^j = \cos(\sqrt{\nu_k}t)U^k\,, \quad \text{respectively}\right).$$

By multiplying both expressions by U^k in the space $\tilde{V}$, and taking Fourier transform from t to β, due to Proposition 4, we have:

$$\sum_{j=1}^\infty \langle U^k\hat{\varphi}^\varepsilon, E_i^\varepsilon\rangle_{\tilde{V}^\varepsilon}\langle U^k, E_i^\varepsilon\rangle_{\tilde{V}} \left(\delta(\beta - \sqrt{\gamma_i^\varepsilon}) + \delta(\beta + \sqrt{\gamma_i^\varepsilon})\right) \underset{\varepsilon\to 0}{\longrightarrow} \delta(\beta - \sqrt{\nu_k})$$
$$+\, \delta(\beta + \sqrt{\nu_k}) \quad \text{in} \quad S'(-\infty,\infty)\,.$$

We proceed exactly as in Sec. VII.12 of Ref. 15 to prove that for ε small there exist eigenvalues $\gamma_i^\varepsilon = \lambda_i^\varepsilon/\varepsilon^{m-2}$ in each small neighborhood of ν_k. Therefore, the theorem is proved. □

5.3. *Global vibrations*

This section is devoted to proving that for each μ_k eigenvalue of the homogenized problem (4.5), the positive roots of the characteristic equation (4.7) can be approached by sequences $\lambda_{i(\varepsilon)}^\varepsilon/\varepsilon^{m-2}$, as $\varepsilon \to 0$, λ_i^ε being eigenvalues of (2.1).

As in Sec. 5.2, we shall also try to obtain spectral properties by knowing properties of some time-depending solutions of vibration problems. But in this case, it is difficult to characterize the problem satisfied by the time-depending limit function. According to Sec. VI.4 of Ref. 16 we introduce Laplace transform to identify this function. At the same time, we use Fourier transforms to have results of spectral convergence. We connect both transforms by using results about boundary values of analytics functions (see Sec. II.2 of Ref. 2).

Let us consider, for each $\lambda < 0$, the stationary problem:

$$\begin{cases} -\Delta u - \alpha^\varepsilon(x)\lambda u = f^\varepsilon & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \Gamma_\Omega \cup \bigcup T^\varepsilon \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \Sigma - \overline{\bigcup T^\varepsilon}\,, \end{cases} \tag{5.15a}$$

where $\alpha^\varepsilon(x) = \rho^\varepsilon(x)\varepsilon^{m-2}, \rho^\varepsilon(x)$ defined in (2.2), and $\{f^\varepsilon\}_\varepsilon \subset V'$ is assumed to be a sequence verifying the relations:

(i) $\|f^\varepsilon\|_{(V^\varepsilon)'} < C$, ($C$ is a constant independent of ε),
(ii) $\exists\, f \in V'$ such that, for each $\phi \in \{u \in C^1(\bar\Omega)/u = 0 \text{ on } \Gamma_\Omega\}$,

$$\langle f^\varepsilon, w^\varepsilon\phi\rangle_{(V^\varepsilon)'\times V^\varepsilon} \longrightarrow \langle f, \phi\rangle_{V'\times V} \quad \text{as} \quad \varepsilon \to 0\,,$$

w^ε being the test functions constructed in Step 1 of Theorem 2, from the solution $U^\lambda(y)$ of the local problem (3.10): $w^\varepsilon = (1 - U^\varepsilon\varphi^\varepsilon)$, $U^\varepsilon(x) = U^\lambda(x/\varepsilon)$. The spaces V^ε and V were defined in (2.3) and (4.6) respectively.

Let us denote by u_λ^ε the unique solution of (5.15a) in the space V^ε satisfying

$$\int_\Omega \nabla u_\lambda^\varepsilon \cdot \nabla v^\varepsilon dx - \lambda \int_\Omega \alpha^\varepsilon(x) u_\lambda^\varepsilon v^\varepsilon dx = \langle f^\varepsilon, v^\varepsilon\rangle_{(V^\varepsilon)'\times V^\varepsilon} \quad \forall\, v^\varepsilon \in V^\varepsilon\,. \tag{5.15b}$$

Proposition 5. *For each $\lambda < 0$ and f^ε satisfying (i) and (ii), u_λ^ε converges weakly in V as $\varepsilon \to 0$, to the unique solution u_λ^0 of the equation:*

$$\int_\Omega \nabla u_\lambda^0 \cdot \nabla v dx - \alpha F(\lambda) \int_\Sigma u_\lambda^0 v d\Sigma = \langle f, v\rangle_{V'\times V} \quad \forall\, v \in V\,. \tag{5.16}$$

Proof. Taking into account (5.15b) and relation (i), we prove that $\{u_\lambda^\varepsilon\}_\varepsilon$ is bounded in $H^1(\Omega)$. Therefore, there exists a subsequence which converges to an element $u_\lambda^0 \in V$ as $\varepsilon \to 0$. The fact that this function is a solution of (5.16) is proved by using the same kind of techniques as in Theorem 2: the test functions are those of property (ii) and because of this property we can pass to the limit in Eq. (5.15b).

On account of relation (3.9), we have $F(\lambda) < 0$ for $\lambda < 0$. So, the uniqueness of solution of (5.16) follows, and the proposition is proved. □

It will prove useful to write, in the sequel, the solution u_λ^0 of (5.16) as the solution of the equation:

$$\left(A - \frac{1}{F(\lambda)} I\right) u_\lambda^0 = \frac{-f}{F(\lambda)} \quad \text{in} \quad V\,, \tag{5.17}$$

A being the compact and self-adjoint operator associated with (4.6) (see Sec. 4). In fact, (5.17) has an unique solution for each $\lambda \in \mathbb{C}$ such that $F(\lambda)$ is not eigenvalue of (4.6) and λ is not a pole of F (see Proposition 2).

Let us consider the hyperbolic problem associated with (2.1) (see also (5.15a)):

$$\begin{cases} \alpha^\varepsilon(x)\dfrac{d^2\mathbf{u}^\varepsilon}{dt^2} + \mathcal{A}^\varepsilon \mathbf{u}^\varepsilon = 0\,, \quad t > 0 \\ u^\varepsilon(0) = 0 \\ \dfrac{d\mathbf{u}^\varepsilon}{dt}(0) = \psi\,, \end{cases} \tag{5.18}$$

$\mathcal{A}^\varepsilon$ being the operator associated with the form defined on V^ε by the left-hand side of (2.3), and $\psi \in C^1(\bar\Omega)$.

From the unitary semigroups theory, there exists a unique generalized solution $\mathbf{u}^\varepsilon$ of (5.18): $\mathbf{u}^\varepsilon \in L^\infty(0,\infty,V^\varepsilon), \mathbf{u}^{\varepsilon'} \in L^\infty(0,\infty,L^2(\Omega))$. The conservation of energy gives us the estimates:

$$\|\mathbf{u}^\varepsilon\|_{L^\infty(0,\infty,V)} \le C \quad \text{and} \quad \|\mathbf{u}^{\varepsilon'}\|_{L^\infty(0,\infty,L^2(\Omega))} \le C$$

(C being a constant independent of ε). Therefore, we can extract a subsequence of $\mathbf{u}^\varepsilon$ such that

$$\begin{cases} \mathbf{u}^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \mathbf{u}^* \text{ in } L^\infty(0,\infty,V) \text{ weak-*, and} \\ \mathbf{u}^{\varepsilon'} \underset{\varepsilon\to 0}{\longrightarrow} \mathbf{u}^{*\prime} \text{ in } L^\infty(0,\infty,L^2(\Omega)) \text{ weak-*}, \end{cases} \tag{5.19}$$

for some $\mathbf{u}^* \in L^\infty(0,\infty,V)$.

In order to identify this element, we use the Laplace transform of $\mathbf{u}^\varepsilon$ and $\mathbf{u}^*$ (see Sec. VI.4 of Ref. 16 relating to this kind of techniques):

$$L[\mathbf{u}^\varepsilon](p) = \int_0^\infty \mathbf{u}^\varepsilon(t)e^{-pt}dt \text{ and } L[\mathbf{u}^*](p) = \int_0^\infty \mathbf{u}^*(t)e^{-pt}dt \text{ for } \mathrm{Re}(p) > 0\,.$$

Now $L[\mathbf{u}^\varepsilon](p)$ and $L[\mathbf{u}^*](p)$ are analytic functions in the half plane $\mathrm{Re}(p) > 0$ with values in V. Besides, from (5.19), we have:

$$L[\mathbf{u}^\varepsilon](p) \underset{\varepsilon\to 0}{\longrightarrow} L[\mathbf{u}^*](p) \text{ weakly in } V\,, \text{ for } \mathrm{Re}(p) > 0\,. \tag{5.20}$$

Proposition 6. *The function $L[\mathbf{u}^*](p)$ is the solution of (5.17) for $\lambda = -p^2$ and $\mathrm{Re}(p) > 0$.*

Proof. By the semigroups theory, $L[\mathbf{u}^\varepsilon](p)$ verifies the equation:

$$\alpha^\varepsilon p^2 L[\mathbf{u}^\varepsilon](p) + \mathcal{A}^\varepsilon L[\mathbf{u}^\varepsilon](p) = \alpha^\varepsilon\psi\,, \qquad L[\mathbf{u}^\varepsilon](p) \in V^\varepsilon \quad \text{for} \quad \mathrm{Re}(p) > 0\,.$$

Therefore, for each real positive p, $L[\mathbf{u}^\varepsilon](p)$ is the solution of (5.15b), being $\lambda = -p^2$ and $f^\varepsilon = \alpha^\varepsilon\psi$. For f^ε, we show relation i) as a consequence of (2.7), and relation ii) by using techniques of type of Step 2 of Theorem 2: we obtain $f = f_\psi(-p^2)$.

$$f_\psi(-p^2) = \alpha \int_B (1 - U^{-p^2}(y))dy \cdot \delta_\Sigma(\psi)\,,$$

where $\delta_\Sigma(\psi)$ is defined as $\langle\delta_\Sigma(\psi),\phi\rangle_{V'\times V} = \int_\Sigma \phi\psi d\Sigma$, and $U^{-p^2}(y)$ is the solution of (3.10) for $\lambda = -p^2$.

Because of Proposition 5, for each real positive p, $L[\mathbf{u}^\varepsilon](p)$ converges to the solution $u^0_{-p^2}$ of (5.16) weakly in V as $\varepsilon \to 0$, with $\lambda = -p^2$ and $f = f_\psi(-p^2)$. Besides, taking into account (5.20) we have: $L[\mathbf{u}^*](p) = u^0_{-p^2}$ for real positive p.

On account of (5.17), $u^0_{-p^2}$ is defined by:

$$u^0_{-p^2} = -\left(A - \frac{1}{F(-p^2)}I\right)^{-1} \frac{f_\psi(-p^2)}{F(-p^2)}, \tag{5.21}$$

and it is an analytic function in $\mathrm{Re}(p) > 0$ which takes the same value as $L[\mathbf{u}^*](p)$ for real positive p. Thus, by analytic continuation, for any p, with positive real part we have: $L[\mathbf{u}^*](p) = u^0_{-p^2}$ and the result of the proposition follows. □

We point out that, by the uniqueness of the inverse Laplace transform and Proposition 6, the convergence of all the sequence $\mathbf{u}^\varepsilon$ to $\mathbf{u}^*$ in $L^\infty(0,\infty,V)$ weak-* as $\varepsilon \to 0$ follows. Let us try to obtain results of spectral convergence from this result by using Fourier transform: we need a result about boundary values, on $\mathrm{Re}(p) = 0$, of analytic functions that we state in the following lemma (see Sec. II.2 of Ref. 2 for its proof):

Lemma 1. *If $f \in S'_+$ ($f \in S'$ with support contained in $[0,+\infty)$), then $L[f](p)$ is an analytic function in the half space $\mathrm{Re}(p) > 0$. Besides, $L[f](\alpha + \mathrm{i}\beta) \longrightarrow \mathcal{F}[f](\beta)$ in S' as $\alpha \to 0^+$, in the sense:*

$$\lim_{\alpha\to 0^+} \langle L[f](\alpha+\mathrm{i}\beta),\ \varphi(\beta)\rangle = \langle \mathcal{F}[f](\beta),\ \varphi(\beta)\rangle \quad \forall\, \varphi \in S'.$$

Let us consider (μ_k, w_k) a fixed eigenelement of (4.5). In the sequel, throughout this section, we shall denote by λ^0 a positive root of the characteristic equation (4.7) (i.e., $F(\lambda^0) = \mu_k$) such that $F'(\lambda^0) \neq 0$ and λ^0 is not an eigenvalue of (3.5). Let us observe that the case when λ^0 is an eigenvalue of (3.5) has been considered in Sec. 5.2. Besides, considering Propositions 2 and 3, for large k there always exists a positive root of (4.7) with $F'(\lambda^0) \neq 0$.

Let $g^\varepsilon(t)$ ($g(t)$ respectively) be the function defined as:

$$g^\varepsilon(t) = \begin{cases} \langle \mathbf{u}^\varepsilon(t), w_k\rangle_V & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \left(g(t) = \begin{cases} \langle \mathbf{u}^*(t), w_k\rangle_V & t \geq 0 \\ 0 & t < 0 \end{cases} \text{ respectively}\right). \tag{5.22}$$

$g^\varepsilon, g \in S'_+$, and on account of Proposition 6 its Laplace transform is given for $\mathrm{Re}(p) > 0$ by:

$$L[g^\varepsilon](p) = L[\langle \mathbf{u}^\varepsilon(t), w_k\rangle_V](p) \ \ (L[g](p) = \langle u^0_{-p^2}, w_k\rangle_V \ \text{ respectively}). \tag{5.23}$$

We define the boundary value of $L[g^\varepsilon](\alpha + \mathrm{i}\beta)$ ($L[g](\alpha+\mathrm{i}\beta)$ respectively) on the imaginary axis $\alpha = 0$ as $\mathcal{F}[g^\varepsilon](\beta)$ ($\mathcal{F}[g](\beta)$ respectively) in the way stated in

Lemma 1. We shall try to identity this distributions in a neighborhood of $\sqrt{\lambda^0}$, by using the Fourier series expansion of $\mathbf{u}^\varepsilon(t)$ (asymptotic expansion of the resolvent operator $(A-\lambda)^{-1}$ in a neighborhood of $\lambda = 1/\mu_k$, respectively).

Proposition 7. *There exists a small neighborhood I of $\sqrt{\lambda^0}$ such that:*

$$\mathcal{F}[g](\beta) = \pi c\delta(\beta - \sqrt{\lambda^0}) - \mathrm{i}P\left(\frac{1}{\beta - \sqrt{\lambda^0}}\right)c + \sum_{n=0}^{\infty}(\mathrm{i}\beta - \sqrt{\lambda^0}\mathrm{i})^n c_n \quad \text{in} \quad D'(I) \tag{5.24}$$

and

$$\mathcal{F}[g^\varepsilon](\beta) = \sum_{i=1}^{\infty} \pi c_i^\varepsilon(\mathrm{i}\beta)\delta(\beta - \sqrt{\mu_i^\varepsilon}) - \mathrm{i}P\left(\frac{1}{\beta - \sqrt{\mu_i^\varepsilon}}\right) c_i^\varepsilon(\mathrm{i}\beta) \quad \text{in} \quad D'(I), \tag{5.25}$$

where c and $c_i^\varepsilon(\mathrm{i}\beta)$ are complex number with $c \neq 0$, $\mathrm{Re}(c) = \mathrm{Re}(c_i^\varepsilon(\mathrm{i}\beta)) = 0$, and $\delta(\beta - a)$($P(\frac{1}{\beta-a})$ respectively) denotes the translated Dirac-Delta (Principal value of $(1/\beta)$ respectively) distribution to the point a. Here μ_i^ε take the value $\lambda_i^\varepsilon/\varepsilon^{m-2}$, with $\{\lambda_i^\varepsilon\}_{i=1}^\infty$ the eigenvalues of (2.1).

Proof. (a) First, we show (5.24). On account of Proposition 6, we consider the formula (5.21) for $u^0_{-p^2}$ and $p = \alpha + \mathrm{i}\beta, \alpha > 0$. Taking into account the Laurent expansion of the resolvent operator $(A-\lambda)^{-1}$ in a reduced neighborhood of $\lambda = 1/\mu_k$, the fact that $F(-p^2)$ and $f_\psi(-p^2)$ are analytic functions in a reduced neighborhood of $p = \pm\sqrt{\lambda^0}\mathrm{i}$, and $F'(\lambda^0) \neq 0$, and Eq. (5.23) for $L[g](p)$, we can write for $\mathrm{Re}(p) > 0$:

$$L[g](p) = c\frac{1}{p - \sqrt{\lambda^0}\mathrm{i}} + \sum_{n=0}^{\infty}(p - \sqrt{\lambda^0}\mathrm{i})^n c_n, \tag{5.26}$$

where c and c_n are constants, c defined as:

$$c = \frac{\mu_k}{F'(\lambda^0)2\mathrm{i}\sqrt{\lambda^0}}\langle \mathcal{P}_k f_\psi(\lambda^0), w_k\rangle_V,$$

$\mathcal{P}_k$ being the projection of the space V on the eigenspace associated with $1/\mu_k$. So, $\mathrm{Re}(c) = 0$.

Let us consider a small interval $I = (\sqrt{\lambda^0} - r, \sqrt{\lambda^0} + r)$ which does not contain other values $\sqrt{\lambda}$, with λ positive root of (4.7) different from λ^0, or $\sqrt{\nu}$, with ν pole of F. Let τ be a smooth function $\tau \in D(I)$. Let us pass to the limit in the relation $\langle L[g](\alpha+\mathrm{i}\beta), \tau(\beta)\rangle_{D'(I)\times D(I)}$ as $\alpha \to 0^+$. On account of (5.26) and Lemma 1, we have

$$\begin{cases} \langle \mathcal{F}[g](\beta), \tau(\beta)\rangle_{D'(I)\times D(I)} \\ \quad = \left\langle \pi c\delta(\beta - \sqrt{\lambda^0}) - \mathrm{i}P\left(\frac{1}{\beta - \sqrt{\lambda^0}}\right)c,\ \tau(\beta)\right\rangle_{D'(I)\times D(I)} \\ \quad + \left\langle \sum_{n=0}^{\infty}(\mathrm{i}\beta - \sqrt{\lambda^0}\mathrm{i})^n c_n,\ \tau(\beta)\right\rangle_{D'(I)\times D(I)} \end{cases} \tag{5.27}$$

(see Sec. II.5 of Ref. 20, relating to the limit of the distribution $1/(\alpha + i\beta)$ as $\alpha \to 0^+$). Therefore, relation (5.24) is proved.

(b) We show (5.25) I being the interval chosen in part (a). Let $\{\mu_i^\varepsilon, e_i^\varepsilon\}_{i=1}^\infty$ be eigenelements of the eigenvalue problem associated with (5.18) (of course $\mu_i^\varepsilon = \lambda_i^\varepsilon/\varepsilon^{m-2}$ with λ_i^ε the eigenvalues of (2.1)), where $\{e_i^\varepsilon\}_{i=1}^\infty$ is assumed to be an orthonormal basis of $L^2(\Omega)$ for the scalar product: $(u,v)_\varepsilon = \int_\Omega \alpha^\varepsilon(x) uv dx$.

We consider the Fourier series expansion of the solution $\mathbf{u}^\varepsilon(t)$ of (5.18) in V^ε:

$$\mathbf{u}^\varepsilon(t) = \sum_{i=1}^\infty \frac{1}{\sqrt{\mu_i^\varepsilon}} (\psi, e_i^\varepsilon)_\varepsilon \sin\left(\sqrt{\mu_i^\varepsilon} t\right) e_i^\varepsilon .$$

Thus, on account of (5.22) and (5.23), we have for each $\tau \in D(I)$:

$$\langle L[g^\varepsilon](\alpha + i\beta), \tau(\beta)\rangle_{D'(I)\times D(I)} = \left\langle -i \sum_{i=1}^\infty c_i^\varepsilon(\alpha + i\beta) \frac{1}{\beta - \sqrt{\mu_i^\varepsilon} - \alpha i}, \tau(\beta) \right\rangle_{D'(I)\times D(I)},$$

where $c_i^\varepsilon(\alpha + i\beta) = \frac{1}{i\beta + \sqrt{\mu_i^\varepsilon} i + \alpha} (\psi, e_i^\varepsilon)_\varepsilon \langle e_i^\varepsilon, w_k\rangle_{\mathsf{V}}$. We pass to the limit in this relation as $\alpha \to 0^+$. Lemma 1 and the same kind of techniques used to prove (5.27) allow us to obtain:

$$\begin{cases} \langle \mathcal{F}[g^\varepsilon](\beta), \tau(\beta)\rangle_{D'(I)\times D(I)} \\ \quad = \left\langle \sum_{i=1}^\infty \pi c_i^\varepsilon(i\beta)\delta(\beta - \sqrt{\mu_i^\varepsilon}) - iP\left(\frac{1}{\beta - \sqrt{\mu_i^\varepsilon}}\right) c_i^\varepsilon(i\beta), \tau(\beta) \right\rangle_{D'(I)\times D(I)}, \end{cases} \tag{5.28}$$

where, we must observe that the number of terms of the summation in which singular distributions appear is finite. Therefore, relation (5.25) is proved. □

Theorem 4. *Let μ_k be an eigenvalue of the homogenized problem (4.6) and λ^0 a positive root of Eq. (4.7) such that $F'(\lambda^0) \neq 0$ and λ^0 is not an eigenvalue of the local problem (3.5). Then λ^0 is a point of accumulation of $\lambda_i^\varepsilon/\varepsilon^{m-2}$ being eigenvalues of (2.1).*

Proof. Because of (5.19) and relation (5.22) we have:

$$\langle \mathcal{F}[g^\varepsilon](\beta), \tau(\beta)\rangle_{D'(I)\times D(I)} \underset{\varepsilon\to 0}{\longrightarrow} \langle \mathcal{F}[g](\beta), \tau(\beta)\rangle_{D'(I)\times D(I)} \tag{5.29}$$

for each $\tau \in D(I)$, I being the interval defined in Proposition 7. By taking imaginary part in (5.27) and (5.28) we obtain from (5.29):

$$\begin{aligned} &\lim_{\varepsilon\to 0} \left\langle -i \sum_{i=1}^\infty \pi c_i^\varepsilon(i\beta)\delta(\beta - \sqrt{\mu_i^\varepsilon}), \tau(\beta) \right\rangle_{D'(I)\times D(I)} \\ &= \langle -i\pi c\delta(\beta - \sqrt{\lambda^0}) + \mathrm{Im}\left(\sum_{n=0}^\infty (i\beta - \sqrt{\lambda^0} i)^n c_n\right), \tau(\beta)\rangle_{D'(I)\times D(I)} . \end{aligned} \tag{5.30}$$

We observe that for each neighborhood J of $\sqrt{\lambda^0}$, $J \subset I$, we can take $\tau \in D(J)$ in such manner that the right-hand side of (5.30) is not zero. So, in each small neighborhood of $\sqrt{\lambda^0}$ we find values $\sqrt{\mu_i^\varepsilon} = \sqrt{\lambda_i^\varepsilon/\varepsilon^{m-2}}$, and the theorem is proved. □

Remark 5. For the extreme cases $\lim_{\varepsilon\to 0} \varepsilon/\eta^2 = 0$ or $\lim_{\varepsilon\to 0} \varepsilon/\eta^2 = \infty$, the result obtained from the formal asymptotical analysis (see Remark 3) is justified by using the techniques of Theorem 2: if $\lambda_{i(\varepsilon)}^\varepsilon/\varepsilon^{m-2} \underset{\varepsilon\to 0}{\longrightarrow} \lambda^0$, and $u_{i(\varepsilon)}^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u^0$ in $H^1(\Omega)$ weakly, and λ^0 is not an eigenvalue of the local problem (3.5), then $u^0 \equiv 0$. In both cases, the result of Theorem 3 follows.

Remark 6. The case when $\varepsilon = \alpha\eta$, with $\alpha \in (0,1)$ has not been considered in this work. The local problem is quite different from that of the case studied here, $\varepsilon = o(\eta)$. Nevertheless, by using the kind of techniques of Proposition 5 and of Sec. II.4 of Ref. 7, we can obtain that if $\lambda_{i(\varepsilon)}^\varepsilon/\varepsilon^{m-2} \underset{\varepsilon\to 0}{\longrightarrow} \lambda^0$, and $u_{i(\varepsilon)}^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u^0$ in $H^1(\Omega)$ weakly, then $u^0 \equiv 0$.

6. Study of The Case $m = 2$

In Sec. 2 we have obtained the estimates (2.5) for the eigenvalues of the problem (2.1) and for $m = 2$. In this section we study the asymptotic behavior of these eigenvalues when $\varepsilon \to 0$. The techniques of formal asymptotic analysis of Sec. 4 allow us to obtain the local problem (3.3) and the homogenized problem:

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega \\ u = 0 \text{ on } \Gamma_\Omega \\ \dfrac{\partial u}{\partial n} = \alpha F(\lambda) u \text{ on } \Sigma \end{cases} \tag{6.1}$$

(F being defined in (3.6) and $\alpha = \lim_{\varepsilon\to 0}(\varepsilon/\eta^2) > 0$) with the spectral parameter appearing both in equation and boundary conditions. The variational formulation of this problem is:

Find λ and $u \in \mathsf{V}, u \neq 0$, satisfying

$$\int_\Omega \nabla u \cdot \nabla v dx = \alpha F(\lambda) \int_\Sigma uv d\Sigma + \lambda \int_\Omega uv dx \quad \forall\, v \in \mathsf{V} \tag{6.2}$$

(V is the space defined in (4.6)).

Problem (6.1) can be considered as an implicit eigenvalue problem for a compact operator $\mathcal{A}(\lambda)$ (see Sec. V. 7 of Ref. 15 for a study of this kind of problems). In fact, the solution of (6.2) is the solution of

$$(\mathcal{A}(\lambda) - I)u = 0 \quad \text{in} \quad \mathsf{V}, \tag{6.3}$$

where $\mathcal{A}(\lambda)$ is defined for real λ:

$$\langle \mathcal{A}(\lambda)u, v\rangle = F(\lambda) \int_\Sigma uv d\Sigma + \lambda \int_\Omega uv dx \quad \forall\, u, v \in \mathsf{V} . \tag{6.4}$$

The family $\{\mathcal{A}(\lambda)\}_{\lambda\in D}$, D being the complex plane without the poles of F, is an analytic family of compacts operators on V (see Proposition 2). Moreover, $\mathcal{A}(\lambda)$ is a self-adjoint operator for real λ, and we have:

Proposition 8. *Problem (6.1) has at most countable number of isolated implicit eigenvalues, with finite multiplicity. Besides these eigenvalues must be positive real numbers.*

Proof. For $\lambda^* < 0$ fixed, $F(\lambda^*) < 0$ (see (3.9)). Therefore $(\mathcal{A}(\lambda^*) - I) \in \mathcal{L}(V)$ is one-to-one, by the open mapping theorem $(\mathcal{A}(\lambda^*) - I)^{-1} \in \mathcal{L}(V)$. On account of Proposition 7.1 of Ref. 15 only a countable number of isolated λ_i, such that 1 is an eigenvalue of $\mathcal{A}(\lambda_i)$ with finite multiplicity, may exist (λ_i is the isolated implicit eigenvalue of (6.1) and a pole of $(\mathcal{A}(\lambda) - I)^{-1}$).

The fact that these eigenvalues, if they exist, are positive real numbers follows from relation (3.9) and relation (6.2). □

We state the result of convergence for $\alpha = \lim_{\varepsilon\to 0}(\varepsilon/\eta^2) > 0$.

Theorem 5. (*a*) *Let $\lambda^{\varepsilon_n}_{i(\varepsilon_n)}$ be a sequence of eigenvalues of (2.1) such that $\lambda^{\varepsilon_n}_{i(\varepsilon_n)}$ converges to λ^0 as $\varepsilon_n \to 0$. Then, λ^0 is an eigenvalue of the local problem (3.5) or an implicit eigenvalue of the homogenized problem (6.1).*
(*b*) *Let λ^0 be an eigenvalue of (3.5), then it is a point of accumulation of λ^ε_i, eigenvalues of (2.1).*
(*c*) *Let λ^0 be an implicit eigenvalue of (6.1) such that the resolvent operator $(\mathcal{A}(\lambda) - I)^{-1}$ has a pole of order 1 in λ^0. Then λ^0 is a point of accumulation of λ^ε_i.*

Proof. Part (a) of the theorem is proved by using the techniques of Sec. 5.1 with minor modifications. Part (b) follows as in Theorem 3 (Sec. 5.2). On the other hand, we can use the techniques of Sec. 5.3 to prove part (c) of the theorem, by introducing convenient modifications. □

6.1. *Extreme cases* $\lim_{\varepsilon\to 0}\varepsilon/\eta^2 = 0$ *and* $\lim_{\varepsilon\to 0}\varepsilon/\eta^2 = \infty, m = 2$

The formal asymptotical analysis give us the local problem (3.3) and the homogenized problem:

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega \\ u = 0 \text{ on } \Gamma_\Omega \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \Sigma \end{cases} \quad \text{for} \quad \lim_{\varepsilon\to 0}\varepsilon/\eta^2 = 0\,, \tag{6.5}$$

and

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \quad \text{for} \quad \lim_{\varepsilon\to 0}\varepsilon/\eta^2 = \infty\,. \tag{6.6}$$

We state the results of convergence for the eigenvalues of (2.1) as $\varepsilon \to 0$, in both cases. These results are quite different from the extreme cases for $m > 2$. Their proof may be performed by using the techniques of Sec. 5, but many calculations are now simplified (see also the techniques of Sec. XII.3 of Ref. 16, relating to the proof of the second part of these results).

Theorem 6. *Let us consider $\lim_{\varepsilon\to 0}\varepsilon/\eta^2 = 0$. Let $\lambda^{\varepsilon_n}_{i(\varepsilon_n)}$ be a sequence of eigenvalues of (2.1) such that $\lambda^{\varepsilon_n}_{i(\varepsilon_n)}$ converges to λ^0 as $\varepsilon_n \to 0$. Then, λ^0 can be an eigenvalue of the local problem (3.5) or an eigenvalue of the homogenized problem (6.5). Reciprocally, if λ^0 is an eigenvalue of problem (3.5) or (6.5) then it is a point of accumulation of eigenvalues of (2.1).*

Theorem 7. *Let us consider $\lim_{\varepsilon\to 0}\varepsilon/\eta^2 = \infty$. Let $\lambda^{\varepsilon_n}_{i(\varepsilon_n)}$ be a sequence of eigenvalues of (2.1) such that $\lambda^{\varepsilon_n}_{i(\varepsilon_n)}$ converges to λ^0 as $\varepsilon_n \to 0$. Then, λ^0 can be an eigenvalue of the local problem (3.5) or an eignevalue of the homogenized problem (6.6). Reciprocally, if λ^0 is an eigenvalue of problem (3.5) or (6.6) then it is a point of accumulation of eigenvalues of (2.1).*

7. Other Boundary Conditions on Σ

In this section we study the asymptotic behavior of the eigenvalues of problem (2.1), as $\varepsilon \to 0$, when the Dirichlet conditions on T^ε are changed by Neumann conditions. Thus, in this case the eigenvalue problem is:

$$\begin{cases} -\Delta u^\varepsilon = \rho^\varepsilon(x)\lambda^\varepsilon u^\varepsilon \text{ in } \Omega \\ u^\varepsilon = 0 \text{ on } \Gamma_\Omega \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 \text{ on } \Sigma\,, \end{cases} \tag{7.1}$$

$\rho^\varepsilon = \rho^\varepsilon(x)$ being the function defined by (2.2) with $m > 2$. The variational formulation of (7.1) is Eq. (2.3), with $u^\varepsilon, v^\varepsilon \in V$ instead of V^ε (cf. (4.6) for definition of V).

Now we can prove estimation (2.5a) (see Proposition 1) for the eigenvalues of (7.1). We use the formal asymptotic expansion as in Sec. 4. On the basis of the expansions (4.1) for a given eigenelement $(\lambda^\varepsilon, u^\varepsilon)$ of (7.1), we obtain the local problem:

$$\begin{cases} -\Delta_y U = \lambda U - \lambda \text{ in } B \\ -\Delta_y U = 0 \text{ in } \mathbb{R}^{3-} - \bar{B} \\ [U] = \left[\dfrac{\partial U}{\partial n}\right] = 0 \text{ on } \Gamma \\ \dfrac{\partial U}{\partial n} = 0 \text{ on } \{y_3 = 0\} \\ U(y) \longrightarrow 0, \quad |y| \to \infty\,,\ y_3 > 0\,, \end{cases} \tag{7.2}$$

and the homogenized problem (4.5) for $\alpha = \lim_{\varepsilon\to 0}(\varepsilon/\eta^2) > 0$. The characteristic equation is now:

$$F^*(\lambda) = \mu_k \tag{7.3}$$

for each μ_k eigenvalue of (7.1), where F^* is the function defined as:

$$F^*(\lambda) = \left\langle \frac{\partial U^\lambda}{\partial n_y}\bigg|_\Gamma, 1 \right\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\,,$$

U^λ being the solution of (7.2). We can study the local problem, and the properties of function F^* in the framework of Sec. 3 (cf. Propositions 2 and 3).

We state the result of convergence for the eigenvalues of (7.1) as follow:

Theorem 8. *(a) Let $\lambda^{\varepsilon_n}_{i(\varepsilon_n)}$ be a sequence of eigenvalues of (7.1) such that $\lambda^{\varepsilon_n}_{i(\varepsilon_n)}/\varepsilon_n^{m-2}$ converges to λ^0 as $\varepsilon_n \to 0$. Then, λ^0 is an eigenvalue of the homogeneous problem associated with (7.2), or $F^*(\lambda^0)$ is an eigenvalue of the homogenized problem (4.5).*
(b) Let λ^0 be an eigenvalue of the homogeneous problem associated with (7.2), then it is a point of accumulation of $\lambda_i^\varepsilon/\varepsilon^{m-2}$ (λ_i^ε being eigenvalues of (7.1)).

Proof. Part (a) of the theorem can be proved by using the same kind of techniques as in Theorem 2: in this case the test functions are constructed taking into account the solution of the local problem (7.2). By using the techniques of Sec. 5.2, with minor modifications, the result sated in (b) follows. □

Remark 7. The case $m < 2$ has not been considered throughout this paper. By analogy with the case in which only a concentrated mass appears (cf. Ref. 10 and Sec. VII.12 of Ref. 15) we can think that the eigenvalues of (2.1) can be approached by those of a problem without concentrated masses. Unlike the precedent case (one concentrated mass), we must consider the effect of the boundary conditions. The asymptotic behavior as $\varepsilon \to 0$, of the eigenvalues of (2.1) can be described via the eigenvalues of the problem:

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega \\ u = 0 \text{ on } \Gamma_\Omega \\ \dfrac{\partial u}{\partial n} = -\alpha C u \text{ on } \Sigma\,, \end{cases}$$

where $C = \int_{\mathbb{R}^3_-} |\nabla W|^2 dy, W(y)$ being the solution of (3.2) (see Sec. V of Ref. 13 relating to this homogenized problem).

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