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VIBRATIONS OF A MEMBRANE WITH MANY CONCENTRATED MASSES NEAR THE BOUNDARY*

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We consider the asymptotic behavior of the vibrations of a membrane occupying a domain $\Omega \subset \mathbb{R}^2$. The density, which depends on a small parameter ε, is of order $O(1)$ out of certain regions where it is $O(\varepsilon^{-m})$ with $m > 0$. These regions, the concentrated masses with diameter $O(\varepsilon)$, are located near the boundary, at mutual distances $O(\eta)$, with $\eta = \eta(\varepsilon) \to 0$. We impose Dirichlet (respectively Neumann) conditions at the points of $\partial\Omega$ in contact with (respectively, out of) the masses. Depending on the value of the parameter m ($m > 2$, $m = 2$ or $m < 2$) we describe the asymptotic behavior of the eigenvalues. Small eigenvalues, of order $O(\varepsilon^{m-2})$ for $m > 2$, are approached via those of a local problem obtained from the micro-structure of the problem, while the eigenvalues of order $O(1)$ are approached through those of a homogenized problem, which depend on the relation between ε and η. Techniques of boundary homogenization and spectral perturbation theory are used to study this problem.

1. Introduction

The study of the vibrations of a membrane containing a small region of diameter $O(\varepsilon)$ including the origin where the density is very much higher than elsewhere has recently been performed by several authors: see Refs. 6, 11 and 12 for a study of the problem using different techniques.

In this paper we deal with the vibrations of a membrane occupying a domain Ω of $\mathbb{R}^2$ that contains many small regions of high density, the so-called *concentrated masses*. These regions share a part of their boundary with that of the boundary of Ω. Moreover, as the size of these small regions decreases, their number increases, as it happens in homogenization problems. This kind of problems for the three-dimensional case have already been considered in Refs. 8 and 9. The results in this paper are quite different from those in Refs. 8 and 9.

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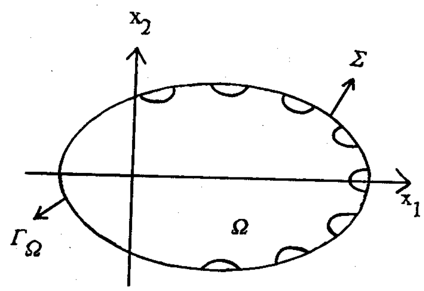

Fig. 1.

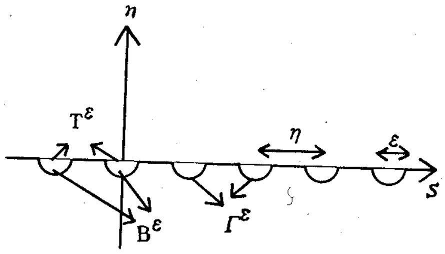

Fig. 2.

We assume that the diameter of the small regions B^ε is $O(\varepsilon)$ and the distance between them is $O(\eta)$, where ε and η are parameters such that $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ (see Fig. 1). The density is of order $O(\varepsilon^{-m})$, $m \geq 0$, in B^ε and $O(1)$ elsewhere. We study the limit behavior, as $\varepsilon \to 0$, of the eigenvalues λ^ε of a mixed problem for the Laplace operator with a Dirichlet condition on $\Sigma \cap \partial B^\varepsilon$ and a Neumann condition on $\Sigma - \partial B^\varepsilon$. This limit behavior depends on the value of the parameter m. The computations and main results do not change very much if we impose a Fourier-type condition on ∂B^ε. Throughout all the paper we assume that Ω has a Lipschitz boundary $\partial\Omega = \bar{\Sigma} \cup \bar{\Gamma}_\Omega$, Σ and Γ_Ω being non-empty parts of the boundary, and Σ is assumed to be a part of a smooth curve. In Secs. 2 and 4 we consider the most general case ($\partial\Omega$ is a smooth curve) while, for simplicity, we perform calculations in Secs. 5 and 6 for the case when Σ is an interval of $\{ (x_1, x_2) \in \mathbb{R}^2 : x_2 = 0 \}$ (see Figs. 1 and 3).

As it is usual in homogenization problems we try to obtain macroscopic information about the eigenfunctions through microscopic information. We obtain microscopic information in a neighborhood of each B^ε by introducing the *local variable* (cf. (4.3)) that enlarges considerably the vicinity of B^ε and leads us to the study of an *eigenvalue local problem* (cf. (3.4)). The study of this problem, as well as of other problems dealing with the microscopic information is performed in Sec. 3.

For $m > 2$, a fact common to this kind of problems with one or several concentrated masses (cf. Refs. 6, 8, 9, 11–13) is that small eigenvalues, of order $O(\varepsilon^{m-2})$, are associated with *local vibrations* while eigenvalues of order $O(1)$ with *global vibrations*. That is to say, when $\lambda^\varepsilon = \lambda^0 \varepsilon^{m-2} + o(\varepsilon^{m-2})$, λ^0 is an eigenvalue of the local problem. The corresponding eigenfunctions u^ε are only significant in a small neighborhood of the concentrated masses (they are approximately eigenfunctions of the local problem associated with λ^0) and they are very small for $|x| = O(1)$. Only for a three-dimensional body containing many concentrated masses near the boundary at mutual distances $\eta \approx \sqrt{\varepsilon}$ (cf. Refs. 8 and 9), can the eigenvalues of order $O(\varepsilon^{m-2})$ give rise to *global vibrations* affecting the whole body.

One might think that results for the three-dimensional body with many concentrated masses could be generalized to the case of a membrane. However, as a consequence of the difference between the behavior at infinity of the functions with a bounded gradient in the plane and those in the space, we prove that the eigenvalues of order $O(\varepsilon^{m-2})$ are only associated with local vibrations (see Secs. 4 and 5.1) while eigenvalues of order $O(1)$ are associated with global vibrations (see Secs. 4 and 5.2). The case $m = 2$ is always a special case: The eigenvalues of order $O(1)$ can cause local and global vibrations (see Refs. 11 and 12 for the *resonance phenomenon* in the case of a single concentrated mass in the membrane).

Using the method of matched asymptotic expansions (see for instance Refs. 8, 9 and 13) we prove in Sec. 4 that expansions $\lambda^\varepsilon = \varepsilon^{m-2}\lambda^0 + O(\varepsilon^{m-2})$ hold when $m > 2$ only for λ^0 an eigenvalue of the local problem. When $m = 2$ the expansions are also possible for λ^0 an eigenvalue either of the local problem or of the *homogenized problem*, which depends on the limit $\alpha = \lim_{\varepsilon\to 0}(-1/(\eta \ln \varepsilon))$ [see (4.11) when $\alpha > 0$, (4.12) when $\alpha = 0$ and (4.13) when $\alpha = +\infty$]. *The critical case*, when $\alpha > 0$, gives us a relation between the size of the B^ε and the distance between them such that the asymptotic behavior of the eigenvalues and the eigenfunctions is different from the behavior of those in *the extreme cases* $\alpha = 0$ and $\alpha = +\infty$. The boundary condition of the homogenized problem is only a consequence of the homogenization of the boundary conditions and not of the concentrated masses. Obviously, for $m > 2$ global vibrations should also exist. We find the corresponding eigenvalues $\lambda^\varepsilon = \lambda^0 + o(1)$ with λ^0 an eigenvalue of the homogenized problem.

Section 5.1 gives the convergence results when λ^0 is an eigenvalue of the local problem. The proofs of these results are based on the Fourier transform method for spectral perturbation problems: we obtain spectral convergence when some properties for the solutions of time-dependent problems are known (see for instance Refs. 8, 9 and 13). We prove the convergence results for the case when λ^0 is an eigenvalue of the homogenized problems in Sec. 5.2 using the Energy Method for boundary homogenization problems (see for instance Refs. 4, 7–9) and the Fourier transform method.

The case $m < 2$ is considered in Sec. 6. Local vibrations are not found. The full mass of the concentrated masses is quite small and these do not influence the vibrations of the membrane. The asymptotic behavior of the eigenvalues is as if the concentrated masses did not exist.

2. Formulation of the Problem

Let Ω be any bounded domain of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$. Let Σ and Γ_Ω be non-empty parts of the boundary, such that $\partial\Omega = \bar{\Sigma} \cup \bar{\Gamma}_\Omega$.

Let us consider orthogonal curvilinear coordinates (s, n) for the point P in a small neighborhood of $\partial\Omega$, where n is the distance between P and $\partial\Omega$ measured inwardly along the unique normal to $\partial\Omega$ passing through P, and s is the length of

$\partial\Omega$ measured anticlockwise between a fixed point on the boundary and the foot of the normal through P to $\partial\Omega$ (see Fig. 1).

Let ε and η be two small parameters such that $\varepsilon < \eta$ and $\eta = \eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let B be the semicircle $B = \{(y_1, y_2) : y_1^2 + y_2^2 < 1,\ y_2 < 0\}$ in the auxiliary space $\mathbb{R}^2$ with coordinates y_1, y_2. Let ∂B be its boundary $\partial B = \bar{T} \cup \bar{\Gamma}$, where T is the interval $T = \{(y_1, 0) : y_1 \in (-1, 1)\}$ lying on the y_2-axis. Let B^ε (and similarly T^ε, Γ^ε) denote its homothetic εB (εT, $\varepsilon\Gamma$) in the s, n variables. Let B_k^ε (and similarly T_k^ε, Γ_k^ε) denote the domain obtained by translation of the previous B^ε (T^ε, Γ^ε) centered on the point $(s_k, 0)$, $s_k = k\eta$, $k \in \mathbb{Z}$ (see Figs. 2 and 3).

Let $\tilde{x}_k$ be the point of Σ of curvilinear coordinates $(s_k, 0)$, for k ranging from $-N(\varepsilon)$ to $N(\varepsilon)$. The number of $\tilde{x}_k$ contained in Σ is $2N(\varepsilon) + 1$ with $N(\varepsilon)$ of order $O(\frac{1}{\eta})$. For each k we define, in the x_1, x_2 variables, the region $B_{\tilde{x}_k}^\varepsilon = \{(x_1(s,n), x_2(s,n)) : (s,n) \in B_k^\varepsilon\}$ and we denote by $T_{\tilde{x}_k}^\varepsilon$ and $\Gamma_{\tilde{x}_k}^\varepsilon$ the parts of its boundary in contact with Σ and contained in Ω respectively (see Fig. 1). We observe that in the x variables $B_{\tilde{x}_k}^\varepsilon$ is no longer a semicircle. Nevertheless, in order to simplify, if there is no ambiguity, we shall also use B^ε (T^ε, Γ^ε) to denote any domain $B_{\tilde{x}_k}^\varepsilon$ ($T_{\tilde{x}_k}^\varepsilon$, $\Gamma_{\tilde{x}_k}^\varepsilon$) for $k \in [-N(\varepsilon), N(\varepsilon)]$.

We study the asymptotic behavior as $\varepsilon \to 0$, of the eigenvalues of the problem:

$$\begin{cases} -\Delta u^\varepsilon = \rho^\varepsilon \lambda^\varepsilon u^\varepsilon \text{ in } \Omega, \\ u^\varepsilon = 0 \text{ on } \Gamma_\Omega \cup \bigcup T^\varepsilon, \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 \text{ on } \Sigma - \overline{\bigcup T^\varepsilon}, \end{cases} \tag{2.1}$$

where $\rho^\varepsilon = \rho^\varepsilon(x)$ is the function defined as:

$$\rho^\varepsilon(x) = \begin{cases} \dfrac{1}{\varepsilon^m}, & \text{if } x \in \bigcup B^\varepsilon \\ 1, & \text{if } x \in \Omega - \overline{\bigcup B^\varepsilon}. \end{cases} \tag{2.2}$$

Here and in the sequel the symbol $\bigcup$ is extended, for fixed ε, to all the regions $B_{\tilde{x}_k}^\varepsilon$, contained in Ω. The parameter m is a real number.

Note that we have considered B as a semicircle for simplicity. Nevertheless, all the results in this paper hold if B is assumed to be an open domain of the lower half-plane $\{(y_1, y_2) \in \mathbb{R}^2 : y_2 < 0\}$ with a Lipschitz boundary, and $T = \partial B \cap \{y_2 = 0\}$ any open interval containing the origin.

We denote by $\mathbf{V}^\varepsilon$ and $\mathbf{V}$ the completion of the spaces $\{u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \Gamma_\Omega \cup \bigcup T^\varepsilon\}$ and of $\{u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \Gamma_\Omega\}$ respectively in the topology of $H^1(\Omega)$.

The variational formulation of problem (2.1) is:

Find λ^ε and $u^\varepsilon \in \mathbf{V}^\varepsilon$, $u^\varepsilon \neq 0$, satisfying the equation

$$\int_\Omega \nabla u^\varepsilon \cdot \nabla v^\varepsilon \, dx = \frac{\lambda^\varepsilon}{\varepsilon^m} \int_{\bigcup B^\varepsilon} u^\varepsilon v^\varepsilon \, dx + \lambda^\varepsilon \int_{\Omega - \overline{\bigcup B^\varepsilon}} u^\varepsilon v^\varepsilon \, dx, \quad \forall\, v^\varepsilon \in \mathbf{V}^\varepsilon. \tag{2.3}$$

For fixed ε, (2.3) is a standard eigenvalue problem. Let

$$0 < \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \cdots \leq \lambda_n^\varepsilon \leq \cdots \xrightarrow{n \to \infty} \infty \tag{2.4}$$

be the sequence of eigenvalues of (2.3), with the classical convention of repeated eigenvalues. Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the corresponding sequence of eigenfunctions, which is assumed to be an orthonormal basis of $\mathbf{V}^\varepsilon$.

Remark 1. It is well known that the classical expression for the Laplacian and for the gradient operators in the curvilinear coordinates s, n (see Sec. II.72 of Ref. 10) are:

$$\begin{cases} \Delta_{sn} = \dfrac{1}{h_s h_n}\left(\dfrac{\partial}{\partial s}\left(\dfrac{h_n}{h_s}\dfrac{\partial}{\partial s}\right) + \dfrac{\partial}{\partial n}\left(\dfrac{h_s}{h_n}\dfrac{\partial}{\partial n}\right)\right), \\ \nabla_{sn} = \left(\dfrac{1}{h_s}\dfrac{\partial}{\partial s}, \dfrac{1}{h_n}\dfrac{\partial}{\partial n}\right), \text{ respectively} \end{cases} \tag{2.5}$$

where $h_s(s,n)$ and $h_n(s,n)$ are smooth functions in the neighborhood of $\partial\Omega$ where the change of variables from x_1, x_2 to s, n are defined. These functions satisfy

$$h_s(s,0) = 1, \qquad h_n(s,n) \equiv 1 \tag{2.6}$$

and the Jacobian of the transformation is the smooth function $h_s(s,n)h_n(s,n)$.

We have the following estimates for the eigenvalues of (2.1).

Proposition 1. (a) *Let us assume $m \geq 2$. For each $i = 1, 2, \ldots, n, \ldots$, we have:*

$$C\varepsilon^{m-2} \leq \lambda_i^\varepsilon \leq C_i \varepsilon^{m-2}. \tag{2.7}$$

(b) *Let us assume $m < 2$. For each $i = 1, 2, \ldots, n, \ldots$, we have:*

$$C \leq \lambda_i^\varepsilon \leq C_i, \tag{2.8}$$

where C and C_i are certain constants, C_i independent of ε, and C independent of ε and i.

Proof. Taking into account Remark 1, for small ε, the elements of $H^1(B_{\bar{x}_k}^\varepsilon)$ can be identified with those of $H^1_{s,n}(B_k^\varepsilon)$ where indices s, n mean that the space is understood in the sense of the Lebesgue measure $ds\,dn$. We consider $v^\varepsilon = u^\varepsilon$ in Eq. (2.3) and then we perform the change of variables from x_1, x_2 to s, n in the integral of (2.3) over the union of all the $B_{\bar{x}_k}^\varepsilon$. Finally, we mainly use the minmax principle and the Poincaré inequality for the elements of $\{u \in H^1_{s,n}(B_k^\varepsilon) : u = 0 \text{ on } T_k^\varepsilon\}$ to obtain (2.7) and (2.8) (see Proposition 1 of Refs. 8 and 9 for this kind of technique). □

Remark 2. The geometric structure of the problem in this section and calculations in Sec. 4 are simplified if Ω is a bounded open domain of $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 < 0\}$ with a Lipschitz boundary $\partial\Omega$ and the part Σ in contact with $\{x_2 = 0\}$ is assumed to be nonempty (see Fig. 3).

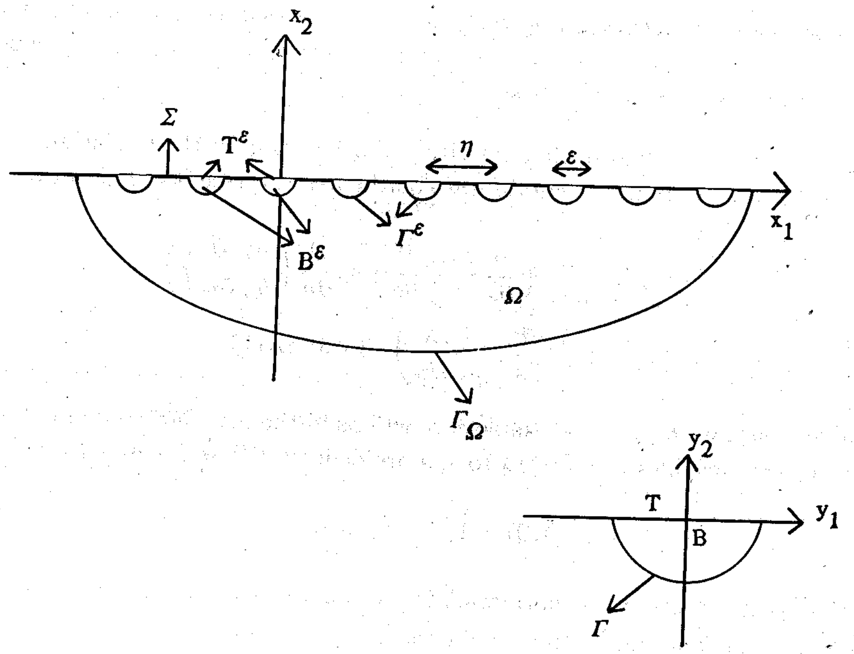

Fig. 3.

3. The Local Problem

As a result of using asymptotic expansions in Sec. 4 we obtain two problems posed in the lower half-plane $\mathbb{R}^{2-} = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 < 0\}$ which, give us microscopic information about the eigenfunctions of (2.1). These problems lead us to the study of an eigenvalue problem, the so-called *local problem*. We study in this section some properties of the solutions of all these problems, and their relation with problem (2.1) will be justified in Sec. 4.

Let us consider $W = W(y)$ the solution of the problem:

$$\begin{cases} -\Delta_y W = 0 \text{ in } \mathbb{R}^{2-}, \\ W = 1 \text{ on } T\,, \\ \dfrac{\partial W}{\partial y_2} = 0 \text{ on } \{y_2 = 0\} - \bar{T}, \\ \dfrac{W(y)}{\ln|y|} \to -\dfrac{1}{\ln 2}, \text{ as } |y| \to \infty,\ y_2 < 0, \end{cases} \tag{3.1}$$

which has already been studied in Ref. 7. The solution in $H^1_{\text{loc}}(\mathbb{R}^{2-})$ is given by:

$$W(y_1, y_2) = -\frac{1}{\pi \ln 2} \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} \ln \sqrt{(y_1 - t)^2 + y_2^2}\, dt\,. \tag{3.2}$$

Let H^λ be the solution of the problem:

$$\begin{cases} -\Delta_y H = \lambda(H + W - 1) \text{ in } B, \\ -\Delta_y H = 0 \text{ in } \mathbb{R}^{2-} - \bar{B}, \\ [H] = \left[\dfrac{\partial H}{\partial n_y}\right] = 0 \text{ on } \Gamma, \\ H = 0 \text{ on } T\ , \\ \dfrac{\partial H}{\partial y_2} = 0 \text{ on } \{y_2 = 0\} - \bar{T}, \\ H(y) \to c, \text{ as } |y| \to \infty,\ y_2 < 0, \end{cases} \tag{3.3}$$

which contains the parameter λ, and where $\bar{n}_y$ denotes the unit outward normal to Γ, the brackets denote the jump across Γ and c is an unknown constant.

The *local problem* is the homogeneous problem associated with (3.3):

$$\begin{cases} -\Delta_y H = \lambda H \text{ in } B, \\ -\Delta_y H = 0 \text{ in } \mathbb{R}^{2-} - \bar{B}, \\ [H] = \left[\dfrac{\partial H}{\partial n_y}\right] = 0 \text{ on } \Gamma, \\ H = 0 \text{ on } T\ , \\ \dfrac{\partial H}{\partial y_2} = 0 \text{ on } \{y_2 = 0\} - \bar{T}, \\ H(y) \to c, \text{ as } |y| \to \infty,\ y_2 < 0\,. \end{cases} \tag{3.4}$$

Provided that λ is not an eigenvalue of (3.4), problem (3.3) has a unique solution $H^\lambda \in \mathcal{V}$, where space $\mathcal{V}$ is the completion of $\{v \in C^1(\bar{B}) : v = 0 \text{ on } T\}$ for the norm of $H^1(B)$. Problem (3.4) amounts to the eigenvalue problem:

Find $\nu \in \mathbb{R}$ and $H \in \mathcal{V}$, $H \neq 0$, satisfying

$$\int_B \nabla H \cdot \nabla V\, dy + \langle \mathcal{T} H|_\Gamma\,,\, V|_\Gamma \rangle = \nu \int_B HV\, dy\ ,\ \forall\, V \in \mathcal{V}, \tag{3.5}$$

where operator $\mathcal{T} \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ is the *normal derivative* operator. As a consequence of the definition of operator $\mathcal{T}$ (see Ref. 6 and Sec. IV.8 of Ref. 13 for definition and properties of $\mathcal{T}$), for each $\phi, \rho \in H^{1/2}(\Gamma)$ the solutions U^ϕ (U^ρ, respectively) of the problem

$$\begin{cases} -\Delta_y U = 0 \text{ in } \mathbb{R}^{2-} - \bar{B}, \\ U = \phi \ \text{ on } \Gamma \quad (U = \rho \text{ on } \Gamma, \text{ respectively})\ , \\ \dfrac{\partial U}{\partial y_2} = 0 \text{ on } \{y_2 = 0\} - \bar{T}, \\ U(y) \to c, \text{ as } |y| \to \infty,\ y_2 < 0\,, \end{cases} \tag{3.6}$$

satisfy

$$\langle \mathcal{T}\phi\,,\, \rho \rangle = \int_{\mathbb{R}^{2-} - \bar{B}} \nabla U^\phi \cdot \nabla U^\rho\, dy.$$

So we have that the form defined by the left-hand side of (3.5) is continuous, symmetric and coercive on $\mathcal{V}$. Therefore, (3.5) is a standard eigenvalue problem having a countable infinity of positive eigenvalues $\{\nu_i\}_{i=1}^{\infty}$.

Remark 3. Taking into account that the solution of (3.6) for $\phi = 1$ is $U^1 \equiv 1$, we have

$$\langle \mathcal{T}H|_\Gamma \,,\, 1|_\Gamma\rangle = \left\langle \frac{\partial H}{\partial n_y} \,,\, 1 \right\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} = 0, \tag{3.7}$$

for H the solution of (3.3) or any eigenfunction of (3.4). This makes a difference in the asymptotic behavior of the eigenvalues of (2.3) between dimensions two and three (see Refs. 8 and 9 for comparison with the bi-dimensional case in this paper).

As it is well known, the behavior of the eigenfunctions of (3.4) and the solution of (3.3), in a neighborhood of infinity is:

$$\begin{cases} H(y) = c + O(|y|^{-1}) \\ \dfrac{\partial H}{\partial y_i}(y) = O(|y|^{-2}), \end{cases} \tag{3.8}$$

while the behavior of the function defined by (3.2) is:

$$\begin{cases} W(y) = O(\ln |y|) \\ \dfrac{\partial W}{\partial y_i}(y) = O(|y|^{-1}). \end{cases} \tag{3.9}$$

For convenience, we introduce here the solutions of two problems (*microscopic problems*) that we obtain using formal asymptotic analysis in Sec. 4: the function $V^\lambda = V^\lambda(y)$ depending on the parameter λ,

$$V^\lambda(y) = 1 - H^\lambda(y) - W(y) \tag{3.10}$$

and the function $U = U(y)$ defined as

$$U(y) = \begin{cases} U^W(y) - W(y), & \text{if } y \in \mathbb{R}^{2-} - B \\ 0, & \text{if } y \in B. \end{cases} \tag{3.11}$$

W and H^λ are the solutions of (3.1) and (3.3) respectively, and $U^W(y)$ is the solution of (3.6) for $\phi = W|_\Gamma$.

4. Asymptotic Expansions for Global Vibrations

In order to have an idea of the asymptotic behavior of the eigen-elements of problem (2.1) we apply the techniques of asymptotic matched expansions. We deal here with *global vibrations* affecting the entire membrane. The results obtained in this section will be justified in Secs. 5 and 6.

Throughout the section we consider $m \geq 2$. We postulate an expansion of the eigenvalues $\lambda^\varepsilon = \lambda_i^\varepsilon$, for a given $i \in \mathbb{N}$:

$$\lambda^\varepsilon = \varepsilon^{m-2}\lambda^0 + \varepsilon^{m-1}\lambda^1 + \cdots , \tag{4.1}$$

and for the corresponding eigenfunctions u^ε an outer expansion in Ω of the form:

$$u^\varepsilon(x) = u^0(x) + \varepsilon u^1(x) + \cdots . \tag{4.2}$$

From (2.1), (4.1) and (4.2) we deduce that u^0 satisfies $-\Delta u^0 = \lambda^0 u^0 \delta_m^2$ in Ω, $u^0 = 0$ on Γ_Ω and some boundary conditions on Σ that will be obtained using the matching relations with local expansions, in an analogous manner to that performed in Refs. 7–9. The symbol δ_m^2 is the Kronecker symbol.

In a neighborhood of Σ we consider curvilinear coordinates (s, n). In order to obtain microscopic information on the eigenfunctions near $x = \tilde{x}_k$, for each fixed k, we introduce the *local variable* y:

$$y_1 = \frac{s - k\eta}{\varepsilon}, \quad y_2 = \frac{n}{\varepsilon}, \tag{4.3}$$

which dilates the neighborhood of each point $(s_k, 0)$, $s_k = k\eta$. $B_{\tilde{x}_k}^\varepsilon$ is B and the closest center to $\tilde{x}_k$ is at distance $O(\eta/\varepsilon)$ in the y variable.

We postulate a local expansion for u^ε of the type:

$$u^\varepsilon = \alpha^0(\varepsilon)v^0(y) + \alpha^1(\varepsilon)v^1(y) + \cdots , \tag{4.4}$$

for some order functions $\alpha^i(\varepsilon)$, $i = 0, 1, 2, \ldots$. Writing (2.1) in the (s, n)-coordinates (see Remark 1) and performing the change of variables from (s, n) to y, we replace $(\lambda^\varepsilon, u^\varepsilon)$ by expansions (4.1), (4.4) in this problem.

On account of (2.6), we obtain that v^0 satisfies the first five equations of (3.4). The condition at infinity,

$$\lim_{\varepsilon \to 0} \alpha^0(\varepsilon) v^0\left(\frac{s - k\eta}{\varepsilon}, \frac{n}{\varepsilon}\right) = u^0(s_k, 0) , \tag{4.5}$$

is obtained from the asymptotic matching principle.

Provided that λ^0 is not an eigenvalue of (3.4), we can write $v^0(y) = u^0(s_k, 0)V^{\lambda^0}(y)$, where V^{λ^0} is the function defined in (3.10) for $\lambda = \lambda^0$. Then, on account of (3.1) and (3.3), relation (4.5) is satisfied for

$$\alpha^0(\varepsilon) = -\frac{\ln 2}{\ln \varepsilon} . \tag{4.6}$$

Function $\alpha^0(\varepsilon)V^{\lambda^0}(y)$ gives us microscopic information about u^ε in a neighborhood of $B_{\tilde{x}_k}^\varepsilon$. Using this information we consider a local expansion for the derivatives of u^ε in a neighborhood of Σ and match it with the outer expansion to obtain:

$$\left.\frac{\partial u^0}{\partial n}\right|_\Sigma = -\lim_{\varepsilon \to 0} \sum_k \frac{\alpha^0(\varepsilon)}{\varepsilon} u^0(s_k, 0) \tau_x \tau_{(s,n)} \left(\frac{\partial V^{\lambda^0}}{\partial n_y}\right)\Bigg|_{\Gamma_{\tilde{x}_k}^\varepsilon} , \tag{4.7}$$

where $\tau_{(s,n)}$ and τ_x denote the change of variables from y to (s,n) and from (s,n) to x, respectively, and the summation is extended to all the centers $\tilde{x}_k$ of $B^\varepsilon_{\tilde{x}_k}$ contained in Σ.

Taking into account Remark 1, (4.3) and (4.6), we calculate the limit on the right-hand side of (4.7) and we obtain the condition satisfied by u^0 on Σ in Cartesian coordinates:

$$\left.\frac{\partial u^0}{\partial n}\right|_\Sigma = -\alpha \ln 2 \left\langle \left.\frac{\partial V^{\lambda^0}}{\partial n_y}\right|_\Gamma, 1 \right\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} u^0|_\Sigma, \tag{4.8}$$

where $\alpha = \lim_{\varepsilon\to 0} \frac{-1}{\eta \ln \varepsilon} > 0$.

Relation $\varepsilon \approx \exp\left(\frac{-1}{\alpha\eta}\right)$ for $\alpha > 0$ gives us *the critical size* of the regions $B^\varepsilon_{\tilde{x}_k}$ such that the asymptotic behavior of u^ε on Σ is a Fourier-type condition, intermediate between Dirichlet and Neumann conditions. In the extreme case $\lim_{\varepsilon\to 0} \frac{-1}{\eta\ln\varepsilon} = 0$ ($\lim_{\varepsilon\to 0} \frac{-1}{\eta\ln\varepsilon} = +\infty$, respectively) we obtain that u^0 satisfies Neumann (Dirichlet, respectively) conditions on Σ.

On account of (3.1), (3.3), (3.7) and (3.10) the right-hand side of (4.8) is given by

$$-\alpha \ln 2 \left(\int_{-1}^{1} \frac{\partial W}{\partial y_2}(y_1, 0)\, dy_1 \right) u^0|_\Sigma$$

and the coefficient of u^0,

$$-\alpha \ln 2 \int_{-1}^{1} \frac{\partial W}{\partial y_2}(y_1, 0)\, dy_1 = -\alpha\pi, \tag{4.9}$$

has already been computed in Ref. 7. Calculations in this section lead us to assert that either value λ^0 is an eigenvalue of the local problem (3.4) or (λ^0, u^0) satisfies

$$\begin{cases} -\Delta u^0 = \lambda^0 u^0 \delta^2_m \text{ in } \Omega, \\ u^0 = 0 \text{ on } \Gamma_\Omega, \\ \dfrac{\partial u^0}{\partial n} = -\alpha\pi u^0 \text{ on } \Sigma. \end{cases} \tag{4.10}$$

That is to say, provided that λ^0 is not an eigenvalue of (3.4), when $m > 2$, the first term of expansion (4.2) is zero for any α. When $m = 2$, u^0 is an eigenfunction associated with λ^0 of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma_\Omega, \\ \dfrac{\partial u}{\partial n} = -\alpha\pi u \text{ on } \Sigma, \end{cases} \quad \text{for } \alpha > 0, \tag{4.11}$$

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma_\Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \Sigma, \end{cases} \quad \text{for } \alpha = 0, \tag{4.12}$$

and

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega, \\ \quad u = 0 \text{ on } \Gamma_\Omega, \quad \text{for } \alpha = +\infty. \\ \quad u = 0 \text{ on } \Sigma, \end{cases} \tag{4.13}$$

Homogenized problems (4.11), (4.12) and (4.13) are standard eigenvalue problems, each one having a countable number of positive eigenvalues converging to infinity. The variational formulation of (4.11) (and (4.12) when $\alpha = 0$) is:

Find $\lambda^0 \in \mathbb{R}$ and $u^0 \in \mathbf{V}$, $u^0 \neq 0$, satisfying

$$\int_\Omega \nabla u^0 \cdot \nabla v \, dx + \alpha\pi \int_\Sigma u^0 v \, d\Sigma = \lambda^0 \int_\Omega u^0 v \, dx, \quad \forall\, v \in \mathbf{V}. \tag{4.14}$$

The variational formulation of (4.13) is:

Find $\lambda^0 \in \mathbb{R}$ and $u^0 \in H_0^1(\Omega)$, $u^0 \neq 0$, satisfying

$$\int_\Omega \nabla u^0 \cdot \nabla v \, dx = \lambda^0 \int_\Omega u^0 v \, dx, \quad \forall\, v \in H_0^1(\Omega). \tag{4.15}$$

Remark 4. The conclusions obtained for $m > 2$ lead us to assert that global vibrations must also exist. Therefore, we look for asymptotic expansions of λ^ε of the type: $\lambda^\varepsilon = \lambda^{0*} + \varepsilon\lambda^{1*} + \cdots$. In this case, we obtain that the microscopic information of u^ε is given by function $\alpha^0(\varepsilon)U(y)$ with U defined in (3.11). We prove that for $\alpha > 0$ ($\alpha = 0$ and $\alpha = +\infty$, respectively) (λ^{0*}, u^0) is an eigen-element of (4.11) ((4.12) and (4.13), respectively), u^0 being the first term arising in expansion (4.2) (cf. Sec. 6.2 of Ref. 9 for a similar situation in dimension three). See Theorem 7 and Remarks 8 and 9 for a study of the critical case $\alpha > 0$, and Theorems 5 and 6 for the cases $\alpha = 0$ and $\alpha = +\infty$.

Remark 5. The results in Remark 4 also hold for $m < 2$. This time, for $\alpha > 0$ we obtain that (λ^{0*}, u^0) is an eigen-element of (4.11), and the microscopic information is given by function $\alpha^0(\varepsilon)(1 - W(y))$, W defined by (3.2).

5. Asymptotic Behavior of the Eigenvalues for $m \geq 2$

For simplicity throughout Secs. 5 and 6 we assume that Ω is a bounded open domain of $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 < 0\}$ with a Lipschitz boundary $\partial\Omega$ and Σ is the part in contact with $\{x_2 = 0\}$ (see Remark 2 and Fig. 3). The proofs of the results stated in both sections can be generalized, with minor modifications, to the general case when Σ is a part of a smooth curve. However, taking into account formula (4.3), calculations become rather awkward and we do not deal with them here.

In this section we consider the case $m \geq 2$ and prove some convergence results for the eigenvalues λ_i^ε of (2.1) for $\varepsilon \to 0$. The results in this section justify to some degree the formal calculations performed in Sec. 4. As in Refs. 8 and 9 we use the Energy Method for boundary homogenization problems and the Fourier transform techniques for time-dependent problems to establish spectral convergence.

We state the main results of the section in the following theorems.

Theorem 1. *Let us consider* $m > 2$ *and* $\alpha = \lim_{\varepsilon\to 0} \frac{-1}{\eta \ln \varepsilon}$. *Each eigenvalue* λ^0 *of the local problem* (3.4) *is a point of accumulation of* $\frac{\lambda_i^\varepsilon}{\varepsilon^{m-2}}$, λ_i^ε *being the eigenvalues of* (2.1). *Besides, if* $\lambda_{i(\varepsilon_n)}^{\varepsilon_n}$ *converges to* λ^0 *as* $\varepsilon_n \to 0$, *then* λ^0 *is an eigenvalue of the homogenized problem* (4.12) *when* $\alpha = 0$ ((4.13) *when* $\alpha = +\infty$). *Reciprocally, when* $\alpha = 0$ ($\alpha = +\infty$ *respectively*) *each eigenvalue of* (4.12) ((4.13) *respectively*) *is a point of accumulation of* λ_i^ε.

Theorem 2. *Let us consider* $m = 2$, $\alpha = \lim_{\varepsilon\to 0} \frac{-1}{\eta \ln \varepsilon}$ *and* λ_i^ε *the eigenvalues of* (2.1). *If* $\lambda_{i(\varepsilon_n)}^{\varepsilon_n}$ *converges to* λ^0 *as* $\varepsilon_n \to 0$, *then* λ^0 *can be an eigenvalue of the local problem* (3.4) *or an eigenvalue of the homogenized problem* (4.11) *when* $\alpha > 0$ ((4.12) *when* $\alpha = 0$ *or* (4.13) *when* $\alpha = +\infty$). *Reciprocally, if* λ^0 *is an eigenvalue of* (3.4) *or an eigenvalue of* (4.11) ((4.12) *and* (4.13) *respectively*) *when* $\alpha > 0$ ($\alpha = 0$ *and* $\alpha = +\infty$ *respectively*), *then* λ^0 *is a point of accumulation of* λ_i^ε.

The proof of these theorems is a direct consequence of Theorem 3 in Sec. 5.1 and Theorems 4–6 in Sec. 5.2.

5.1. *Local vibrations*

We introduce two hyperbolic problems for special initial data associated with (2.1) and (3.4) respectively and use the Fourier transform to prove that each eigenvalue of the local problem (3.4) can be approached by sequences $\frac{\lambda_{i(\varepsilon)}^\varepsilon}{\varepsilon^{m-2}}$, as $\varepsilon \to 0$. Once we have chosen suitable initial conditions for the hyperbolic problems, the proof of the convergence is quite standard. Thus, we only sketch here the proof (see Sec. 5.2 of Ref. 8 and Sec. VII.12 of Ref. 13 for this kind of technique).

Let us change the variable in (2.3) by setting $y = x/\varepsilon$. We obtain:

$$\int_{\Omega_\varepsilon} \nabla_y U^\varepsilon \cdot \nabla_y V^\varepsilon \, dy = \gamma^\varepsilon \int_{\Omega_\varepsilon} \beta^\varepsilon(y) U^\varepsilon V^\varepsilon \, dy, \quad \forall\ V^\varepsilon \in \tilde{\mathbf{V}}^\varepsilon, \tag{5.1}$$

Ω_ε being the space $\{y : \varepsilon y \in \Omega\}$, $\gamma^\varepsilon = \lambda^\varepsilon/\varepsilon^{m-2}$, and $\beta^\varepsilon(y)$ defined as:

$$\beta^\varepsilon(y) = \begin{cases} 1, & \text{if } y \in \bigcup \tau_y B^\varepsilon \\ \varepsilon^m, & \text{if } y \in \Omega_\varepsilon - \overline{\bigcup \tau_y B^\varepsilon}, \end{cases}$$

where $\tau_y B^\varepsilon$ denote the transformed domains of the regions B^ε contained in Ω to the y variable. $\tilde{\mathbf{V}}^\varepsilon$ is the functional space $\{U = U(y) : U(\varepsilon y) \in \mathbf{V}^\varepsilon\}$. Let $\{\gamma_j^\varepsilon, E_j^\varepsilon\}_{j=1}^\infty$ be the eigen-elements of (5.1), where $\{E_j^\varepsilon\}_{j=1}^\infty$ is assumed to be an orthonormal basis of $\tilde{\mathbf{V}}^\varepsilon$.

For convenience we write the variational formulation of (3.4) in the form: Find $\nu > 0$ and $U \in \tilde{\mathcal{V}}$, $U \neq 0$, satisfying

$$\int_{\mathbb{R}^{2-}} \nabla_y U \cdot \nabla_y V \, dy = \nu \int_B UV \, dy \,, \ \forall\ V \in \tilde{\mathcal{V}}, \tag{5.2}$$

where $\tilde{\mathcal{V}}$ is the completion of $\{U \in \mathcal{D}(\overline{\mathbb{R}^{2-}}) : U = 0 \text{ on } T\}$ for the Dirichlet norm $\|\nabla_y U\|_{L^2(\mathbb{R}^{2-})}$. It is evident that $\tilde{\mathcal{V}} \subset L^2(B)$ with dense and compact embedding. Let $\{\nu_j\,,\, U^j\}_{j=1}^{\infty}$ be the eigen-elements of (5.2) with $\{U^j\}_{j=1}^{\infty}$ an orthonormal basis of $\tilde{\mathcal{V}}$.

For each k, we consider $(\nu_k\,,\, U^k)$ a fixed eigen-element of (5.2). Let $\tilde{\varphi}^\varepsilon(y)$ be the function defined as:

$$\tilde{\varphi}^\varepsilon(y) = \begin{cases} 1, & \text{if } |y| \le R_\varepsilon \\ 1 - \dfrac{\ln|y| - \ln R_\varepsilon}{\ln R_\varepsilon}, & \text{if } R_\varepsilon \le |y| \le R_\varepsilon^2 \\ 0, & \text{if } |y| \ge R_\varepsilon^2, \end{cases} \tag{5.3}$$

where $R_\varepsilon = \sqrt{\frac{\varepsilon+\eta/4}{\varepsilon}}$. On account of $\varepsilon = o(\eta)$, formula (5.3) and the fact that $U^k \in \tilde{\mathcal{V}}$ is a harmonic function outside $\overline{B(0,1)}$ and satisfies (3.8) for large values of $|y|$, we show: $\tilde{\varphi}^\varepsilon \in H_0^1(B(0, R_\varepsilon^2))$, $U^k\tilde{\varphi}^\varepsilon \in \tilde{\mathbf{V}}^{\varepsilon}$,

$$\|\nabla_y (U^k \tilde{\varphi}^\varepsilon)\|_{L^2(\mathbb{R}^{2-})} \le \text{Cte} \tag{5.4}$$

and

$$U^k \tilde{\varphi}^\varepsilon \xrightarrow{\varepsilon \to 0} U^k \text{ in } \tilde{\mathcal{V}}. \tag{5.5}$$

We observe that Cte in (5.4) is a constant independent of ε and the elements of $\tilde{\mathbf{V}}^{\varepsilon}$ prolonged by zero in $\mathbb{R}^{2-} - \overline{\Omega_\varepsilon}$ are elements of $\tilde{\mathcal{V}}$.

Let $\mathbf{U}^{\varepsilon} \in L^\infty(0, \infty, \tilde{\mathbf{V}}^{\varepsilon})$ be the generalized solution of the hyperbolic problem corresponding to (5.1):

$$\begin{cases} \beta^{\varepsilon}(y) \dfrac{d^2 \mathbf{U}^{\varepsilon}}{dt^2} + \mathcal{A}^{\varepsilon}\, \mathbf{U}^{\varepsilon} = 0, \quad t > 0, \\ \mathbf{U}^{\varepsilon}(0) = U^k \tilde{\varphi}^\varepsilon, \\ \dfrac{d\mathbf{U}^{\varepsilon}}{dt}(0) = 0, \end{cases} \tag{5.6}$$

where $\mathcal{A}^{\varepsilon}$ is the operator associated with the form defined on $\tilde{\mathbf{V}}^{\varepsilon}$ by the left-hand side of (5.1). Writing the formula of the conservation of the energy for $\mathbf{U}^{\varepsilon}(t)$, (5.4) leads us to assert that there is a function $\mathbf{U} \in L^\infty(0, \infty, \tilde{\mathcal{V}})$ such that some subsequence $\mathbf{U}^{\varepsilon}$ satisfies:

$$\begin{cases} \mathbf{U}^{\varepsilon} \xrightarrow{\varepsilon \to 0} \mathbf{U} \text{ in } L^\infty(0, \infty, \tilde{\mathcal{V}}) \text{ weak} - *, \text{ and} \\ \dot{\mathbf{U}}^{\varepsilon} \xrightarrow{\varepsilon \to 0} \dot{\mathbf{U}} \text{ in } L^\infty(0, \infty, L^2(B)) \text{ weak} - *. \end{cases} \tag{5.7}$$

On account of (5.4) and (5.5) we identify $\mathbf{U}$ in (5.7) with the generalized solution of the hyperbolic problem corresponding to (5.2):

$$\begin{cases} \beta(y) \dfrac{d^2 \mathbf{U}}{dt^2} + \mathcal{A}\mathbf{U} = 0, \quad t > 0, \\ \mathbf{U}(0) = U^k, \\ \dfrac{d\mathbf{U}}{dt}(0) = 0, \end{cases} \tag{5.8}$$

$\beta(y)$ being the function defined as 1 in B and 0 in $\mathbb{R}^{2-} - \bar{B}$, and $\mathcal{A}$ the operator associated with the form defined by the left-hand side of (5.2).

Now the spectral convergence holds as stated in the following theorem.

Theorem 3. *Each eigenvalue ν_k of the local problem (3.4) is a point of accumulation of $\frac{\lambda_i^\varepsilon}{\varepsilon^{m-2}}$, λ_i^ε being the eigenvalues of (2.1) and $m \geq 2$.*

Proof. Let us consider the Fourier series expansion of the solution $\mathbf{U}^\varepsilon$ of (5.6) ($\mathbf{U}$ of (5.8), respectively) in terms of $\{E_j^\varepsilon\}_{j=1}^\infty$ in $\tilde{\mathbf{V}}^\varepsilon$ ($\{U^j\}_{j=1}^\infty$ in $\tilde{\mathcal{V}}$, respectively). We multiply both expansions by U^k in the space $\tilde{\mathcal{V}}$, and take the Fourier transform from t to β. On account of (5.7), we have:

$$\sum_{j=1}^{\infty} c_j^\varepsilon \left[\delta\left(\beta - \sqrt{\gamma_j^\varepsilon}\right) + \delta\left(\beta + \sqrt{\gamma_j^\varepsilon}\right)\right] \xrightarrow{\varepsilon \to 0} \delta(\beta - \sqrt{\nu_k}) + \delta(\beta + \sqrt{\nu_k})$$

in $\mathcal{S}'(-\infty, \infty)$, where $c_j^\varepsilon = \langle U^k \tilde{\varphi}^\varepsilon, E_j^\varepsilon \rangle_{\tilde{\mathbf{V}}^\varepsilon} \langle U^k, E_j^\varepsilon \rangle_{\tilde{\mathcal{V}}}$. Then, the result of the theorem holds (see Theorem 3 of Ref. 8, for example). □

5.2. *Global vibrations*

This section is devoted to proving the results of Theorems 1 and 2 not dealing with the eigenvalues of the local problem. Results when $m > 2$ are qualitatively different from the case when $m = 2$, as announced in Sec. 1. Nevertheless, in the proofs of Theorems 4 and 6, we consider both cases together because the reasoning is of the same type.

It will be useful to introduce here the *cutoff functions* φ^ε that we shall use to construct suitable test functions in order to prove the results in this, and the next, sections (see Refs. 2, 8 and 9 for similar constructions). Let φ^ε be a smooth function which takes the value 1 in the semicircle of radius $(\varepsilon + \eta/8)$, $B(\varepsilon + \eta/8)$, and is zero outside the semicircle of radius $(\varepsilon + \eta/4)$, $B(\varepsilon + \eta/4)$:

$$\varphi^\varepsilon(x) = \varphi\left(2\frac{|x| - \varepsilon}{\eta}\right),$$

where $\varphi \in C^\infty[0,1]$, $0 \leq \varphi \leq 1$, $\varphi = 1$ in $[0, 1/4]$ and $\mathrm{Supp}(\varphi) \subset [0, 1/2]$.

We denote by $\tilde{W}^\varepsilon$ the function

$$\tilde{W}^\varepsilon(x) = 1 + \frac{\ln 2}{\ln \varepsilon}\left(1 - W\left(\frac{x}{\varepsilon}\right)\right), \tag{5.9}$$

where W is the solution of (3.1). On account of (3.2) and (3.9) it is easy to prove:

$$\begin{cases} \tilde{W}^\varepsilon(x) \leq \mathrm{Cte}\,\dfrac{1 + |\ln \eta|}{|\ln \varepsilon|} & \text{for } x \in C^\varepsilon, \\ \dfrac{\partial \tilde{W}^\varepsilon}{\partial x_i}(x) \leq \dfrac{\mathrm{Cte}}{\eta |\ln \varepsilon|} & \text{for } x \in C^\varepsilon, \end{cases} \tag{5.10}$$

C^ε being the region $C^\varepsilon = B(\varepsilon+\eta/4) - \overline{B(\varepsilon+\eta/4)}$ and Cte a constant independent of ε.

Let us consider the functions w^ε defined as

$$w^\varepsilon = 1 - \tilde{W}^\varepsilon \varphi^\varepsilon \quad \text{in } B(\varepsilon+\eta/4) \tag{5.11}$$

and prolonged by periodicity over all the regions $B(\varepsilon+\eta/4)$ centered on $\tilde{x}_k$ contained in Ω, and by value 1 outside.

In an analogous way we define function $\tilde{w}^\varepsilon$

$$\tilde{w}^\varepsilon = 1 - \tilde{U}^\varepsilon \varphi^\varepsilon \quad \text{in } B(\varepsilon+\eta/4)\,, \tag{5.12}$$

$\tilde{U}^\varepsilon$ now being $\tilde{U}^\varepsilon = \left(1 + \frac{\ln 2}{\ln\varepsilon} U(\frac{x}{\varepsilon})\right)$ and with U defined by (3.11). $\tilde{U}^\varepsilon$ also satisfies estimates (5.10).

On account of (4.4)–(4.6), Remarks 4 and 5 and estimates (3.8), w^ε and $\tilde{w}^\varepsilon$ are used as *test functions* in the proof of the following theorems. We state the main properties of these functions in Propositions 2 and 3.

Proposition 2. *Let us consider* $\alpha = \lim_{\varepsilon\to 0} \frac{-1}{\eta \ln\varepsilon} \geq 0$. *The sequence* w^ε *defined in* (5.11) *satisfy:*

(a) *For fixed* $\varepsilon > 0$, $w^\varepsilon \in H^1(\Omega)$, $w^\varepsilon = 0$ *on* $\bigcup T^\varepsilon$, *and takes the value* $\left(-\frac{\ln 2}{\ln\varepsilon}\left(1 - W(\frac{x-\tilde{x}_k}{\varepsilon})\right)\right)$ *in* $B^\varepsilon_{\tilde{x}_k}$, *for each* $k \in [-N(\varepsilon), N(\varepsilon)]$.

(b) $\int_\Omega |\nabla w^\varepsilon|^2\,dx \leq C$ *(constant* C *does not depend on* ε*), and* $w^\varepsilon \xrightarrow{\varepsilon\to 0} 1$ *in* $H^1(\Omega)$*-weak.*

(c) $\int_\Omega |\nabla w^\varepsilon|^2 \phi\,dx \xrightarrow{\varepsilon\to 0} \alpha\pi \int_\Sigma \phi\,d\Sigma$ *for each function* ϕ, $\phi \in C(\bar{\Omega})$.

(d) *For each* $v \in \mathbf{V}$ *and each sequence* $\{v^\varepsilon\}_\varepsilon$ *with* $v^\varepsilon \in \mathbf{V}^\varepsilon$ *and* $v^\varepsilon \xrightarrow{\varepsilon\to 0} v$ *weakly in* $H^1(\Omega)$, *there exists a sequence* $\tilde{v}^\varepsilon \in H^1(\Omega)$ *such that* $\tilde{v}^\varepsilon = 0$ *on* $\Gamma_\Omega \cup \bigcup T^\varepsilon$, $\tilde{v}^\varepsilon \xrightarrow{\varepsilon\to 0} v$ *weakly in* $H^1(\Omega)$ *and*

$$\int_\Omega \nabla w^\varepsilon \cdot \nabla \tilde{v}^\varepsilon\,\phi\,dx \xrightarrow{\varepsilon\to 0} \alpha\pi \int_\Sigma v\phi\,d\Sigma. \tag{5.13}$$

Besides,

$$\frac{1}{\varepsilon^2} \int_{\bigcup B^\varepsilon} w^\varepsilon v^\varepsilon \phi\,dx \xrightarrow{\varepsilon\to 0} 0, \tag{5.14}$$

ϕ *in* (5.13) *and* (5.14) *being any function of* $\{u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \Gamma_\Omega\}$.

Proof. Property (a) is a consequence of the definition of φ^ε and $\tilde{W}^\varepsilon$. On account of (5.10), (5.11) and estimates for φ^ε and its derivatives we prove:

$$\int_\Omega |\nabla w^\varepsilon|^2\,dx = \int_{\bigcup B(\varepsilon+\eta/8)} |\nabla \tilde{W}^\varepsilon|^2\,dx + o(1),$$

with $o(1) \to 0$ as $\varepsilon \to 0$. Now, by applying the Green formula on each $B(\varepsilon + \eta/8)$ and taking (5.9), (5.10), (3.1), (3.2) and (4.9) into account we prove statement (b), as well as (c), for any step function ϕ. So, (c) is also true for $\phi \in C(\bar{\Omega})$.

For each fixed $h > 0$, we consider a regular triangulation of the domain Ω of diameter h. Let $v^{\varepsilon,h}$ be the product of w^ε by the projection of v^ε on the space of the continuous functions over $\bar{\Omega}$ which are polynomials of degree 1 on each triangle and take the value zero on Γ_Ω. On account of (c) and by a process of taking limits first as $\varepsilon \to 0$, and later as $h \to 0$ in the integral

$$\int_\Omega \nabla w^\varepsilon \cdot \nabla v^{\varepsilon,h} \, \phi \, dx$$

we may choose a sequence $h(\varepsilon)$, with $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and with the functions $\tilde{v}^\varepsilon = v^{\varepsilon,h(\varepsilon)}$ converging to v weakly in $H^1(\Omega)$ and satisfying (5.13) (see Sec. I.3.4 of Refs. 1, 5.1 of Refs. 8 and Ref. 2 dealing with the techniques used for this proof). (5.14) is a consequence of property (a) and the Poincaré inequality on each region $B^\varepsilon_{\bar{x}_k}$. Therefore the statements in proposition hold. □

Proposition 3. *Let us consider* $\alpha = \lim_{\varepsilon \to 0} \frac{-1}{\eta \ln \varepsilon} \geq 0$. *Functions* $\tilde{w}^\varepsilon$ *defined in* (5.12) *satisfy:*

(a) *For fixed* $\varepsilon > 0$, $\tilde{w}^\varepsilon \in H^1(\Omega)$, $\tilde{w}^\varepsilon = 0$ *in* $\bigcup B^\varepsilon$.

(b) $\int_\Omega |\nabla \tilde{w}^\varepsilon|^2 \, dx \leq C$ *(constant* C *does not depend on* ε*), and* $\tilde{w}^\varepsilon \xrightarrow{\varepsilon \to 0} 1$ *in* $H^1(\Omega)$*-weak.*

(c) $\int_\Omega |\nabla \tilde{w}^\varepsilon|^2 \phi \, dx \xrightarrow{\varepsilon \to 0} \alpha c^* \int_\Sigma \phi \, d\Sigma$, *for each function* ϕ, $\phi \in C(\bar{\Omega})$ *(*c^* *is the constant,* $c^* = -\ln 2 \, \langle \frac{\partial U}{\partial n_y}, 1 \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}$ *and* U *defined by* (3.11)*).*

(d) *For each* $v \in \mathbf{V}$ *and each sequence* $\{v^\varepsilon\}_\varepsilon$ *with* $v^\varepsilon \in \mathbf{V}^\varepsilon$ *and* $v^\varepsilon \xrightarrow{\varepsilon \to 0} v$ *weakly in* $H^1(\Omega)$*, there exists a sequence* $\tilde{v}^\varepsilon \in H^1(\Omega)$ *such that* $\tilde{v}^\varepsilon = 0$ *on* $\Gamma_\Omega \cup \bigcup B^\varepsilon$, $\tilde{v}^\varepsilon \xrightarrow{\varepsilon \to 0} v$ *weakly in* $H^1(\Omega)$ *and*

$$\int_\Omega \nabla \tilde{w}^\varepsilon \cdot \nabla \tilde{v}^\varepsilon \, \phi \, dx \xrightarrow{\varepsilon \to 0} \alpha c^* \int_\Sigma v \phi \, d\Sigma, \tag{5.15}$$

ϕ *being any function of* $\{u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \Gamma_\Omega\}$.

Proof of Proposition 3 holds as 2 and it is not given here.

Remark 6. On account of (c) in Propositions 2 and 3 we observe that in the case when $\lim_{\varepsilon \to 0} \frac{-1}{\eta \ln \varepsilon} = 0$ the convergence of w^ε and $\tilde{w}^\varepsilon$ to 1, as $\varepsilon \to 0$, takes place in the strong topology of $H^1(\Omega)$. Besides, considering that (5.10) holds for $\varepsilon < \eta$, in the case when $\lim_{\varepsilon \to 0} \frac{-1}{\eta \ln \varepsilon} = +\infty$ we multiply the terms depending on ε in (c), (5.13), (5.14) and (5.15) by $(-\eta \ln \varepsilon)$, and we prove that these new sequences converge, as $\varepsilon \to 0$, towards the same limit, where we now have identified α with 1.

Theorem 4. *Let* m *be* ≥ 2*, and* $\alpha = \lim_{\varepsilon \to 0} \frac{-1}{\eta \ln \varepsilon}$. *Let* $(\lambda_i^\varepsilon, u_i^\varepsilon)$ *be the eigenelements of* (2.1)*. Let us suppose that there is a sequence* $\varepsilon_n \to 0$ *such that:*

$$\frac{\lambda_i^{\varepsilon_n}}{\varepsilon_n^{m-2}} \xrightarrow{n\to\infty} \lambda^0,$$

$$u_i^{\varepsilon_n} \xrightarrow{n\to\infty} u^0 \quad \textit{weakly in } H^1(\Omega),$$

where the index i may depend on ε_n. *Then,*

(i) *if* $m > 2$, $u^0 \equiv 0$,
(ii) *if* $m = 2$ *and* $u^0 \neq 0$, (λ^0, u^0) *is an eigen-element of* (4.11) *when* $\alpha > 0$. ((4.12) *when* $\alpha = 0$ *and* (4.13) *when* $\alpha = +\infty$, *respectively*).

Proof. Let us first consider the case when $\alpha = \lim_{\varepsilon\to 0} \frac{-1}{\eta \ln \varepsilon} \geq 0$. For simplicity we write $(\lambda^\varepsilon, u^\varepsilon)$ instead of $(\lambda_i^{\varepsilon_n}, u_i^{\varepsilon_n})$. Statement (a) in Proposition 2, allow us to take $v^\varepsilon = \phi w^\varepsilon$ in (2.3):

$$\int_\Omega \nabla u^\varepsilon \cdot \nabla(w^\varepsilon \phi)\, dx = \frac{\lambda^\varepsilon}{\varepsilon^{m-2}} \frac{1}{\varepsilon^2} \int_{\bigcup B^\varepsilon} u^\varepsilon w^\varepsilon \phi\, dx + \lambda^\varepsilon \int_{\Omega - \overline{\bigcup B^\varepsilon}} u^\varepsilon w^\varepsilon \phi\, dx, \quad (5.16)$$

with $\phi \in \{u \in C^1(\bar\Omega) : u = 0 \text{ on } \Gamma_\Omega\}$ and w^ε defined in (5.11). Taking properties (b) and (d) for w^ε into account and considering the sequence $\{\tilde u^\varepsilon\}_\varepsilon$ satisfying (5.13), we obtain:

$$\int_\Omega \nabla u^0 \cdot \nabla\phi\, dx + \alpha\pi \int_\Sigma u^0 \phi d\Sigma = \delta_m^2 \lambda^0 \int_\Omega u^0 \phi\, dx - \lim_{\varepsilon\to 0} \int_\Omega \nabla(u^\varepsilon - \tilde u^\varepsilon) \cdot \nabla w^\varepsilon \phi\, dx. \quad (5.17)$$

As $u^\varepsilon = \tilde u^\varepsilon = 0$ on $\bigcup T^\varepsilon$, similar reasoning to that performed in the proof of (b) in Proposition 2 leads us to assert that the limit occurring in (5.17) is 0. On account of (4.14), the results stated in (i) and (ii) of the theorem are true for $\alpha \geq 0$.

To prove the assertions of (i) and (ii) when $\alpha = +\infty$, we observe that (λ^0, u^0) satisfies $u^0 = 0$ on Γ_Ω, and equation $-\Delta u^0 = \delta_m^2 \lambda^0 u^0$ in $\mathcal{D}'(\Omega)$. Taking Remark 6 into account, the condition $u^0 = 0$ on Σ is deduced by multiplying (5.16) by $(\eta \ln \varepsilon)$ and taking limits when $\varepsilon \to 0$; we obtain:

$$\int_\Sigma u^0 \phi\, d\Sigma = 0, \quad \forall\, \phi \in \{u \in C^1(\bar\Omega) : u = 0 \text{ on } \Gamma_\Omega\}.$$

Therefore, the theorem is proved. □

Theorem 5. *Let m be* > 2, *and* $\alpha = \lim_{\varepsilon\to 0} \frac{-1}{\eta \ln \varepsilon}$. *Let* $(\lambda_i^\varepsilon, u_i^\varepsilon)$ *be the eigen-elements of* (2.1). *Let us suppose that there is a sequence* $\varepsilon_n \to 0$ *such that:*

$$\lambda_{i(\varepsilon_n)}^{\varepsilon_n} \xrightarrow{n\to\infty} \lambda^0$$

$$u_{i(\varepsilon_n)}^{\varepsilon_n} \xrightarrow{n\to\infty} u^0 \quad \textit{weakly in } H^1(\Omega),$$

where u^0 *is assumed to be nonzero. Then,* (λ^0, u^0) *is an eigen-element of* (4.12) *when* $\alpha = 0$ ((4.13) *when* $\alpha = +\infty$, *respectively*).

Proof. We rewrite the proof of Theorem 4 replacing w^ε by the test functions $\tilde{w}^\varepsilon$ in (5.12) and taking into account Proposition 3 and Remark 6. □

The following theorem gives us some complementary results to those in Theorems 4 and 5. We show the theorem using the techniques in Sec. 5.1.

Theorem 6. *Let us consider* $\alpha = \lim_{\varepsilon\to 0} \frac{-1}{\eta \ln \varepsilon}$.

(i) *Let* m *be equal to 2 and* λ^0 *be an eigenvalue of* (4.11) *for* $\alpha > 0$. ((4.12) *for* $\alpha = 0$ *and* (4.13) *for* $\alpha = +\infty$, *respectively), then it is a point of accumulation of* λ_i^ε (λ_i^ε *eigenvalues of* (2.1)).

(ii) *Let* m *be* > 2 *and* λ^0 *be an eigenvalue of* (4.12) *for* $\alpha = 0$ ((4.13) *for* $\alpha = +\infty$, *respectively), then it is a point of accumulation of* λ_i^ε.

Proof. As in Sec. 5.1, once we have chosen suitable initial data for some hyperbolic problems associated with (2.1), the proof of the theorem is quite standard in both cases (i) and (ii) (see Theorem 7 of Ref. 9 for a similar proof). We only write here the hyperbolic problem depending on ε.

Let us consider the hyperbolic problem associated with (2.1),

$$\begin{cases} \rho^\varepsilon(x)\dfrac{d^2\mathbf{u}^\varepsilon}{dt^2} + \mathcal{A}^\varepsilon \mathbf{u}^\varepsilon = 0, \quad t > 0, \\ \mathbf{u}^\varepsilon(0) = r^\varepsilon, \\ \dfrac{d\mathbf{u}^\varepsilon}{dt}(0) = 0, \end{cases} \tag{5.18}$$

where $\mathcal{A}^\varepsilon$ is the operator associated with the form defined on $\mathbf{V}^\varepsilon$ by the left-hand side of (2.3). The initial data $(r^\varepsilon, 0) \in \mathbf{V}^\varepsilon \times L^2(\Omega)$ for the proof of (i) ((ii), respectively) are given by $r^\varepsilon = \psi w^\varepsilon$ ($r^\varepsilon = \psi\tilde{w}^\varepsilon$, respectively) with $\psi \in \{u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \Gamma_\Omega\}$ such that $(\psi, u^0)_{L^2(\Omega)} \neq 0$, and w^ε the function in (5.11) ($\tilde{w}^\varepsilon$ in (5.12), respectively). Finally we use Proposition 2 and Theorem 4 (3 and 5, respectively) to identify the limit of $\mathbf{u}^\varepsilon(t)$ in $L^\infty(0,\infty,\mathbf{V})$ weak-* and to obtain the assertions of the theorem. □

Remark 7. In the proofs of Theorems 4 and 6, function $\tilde{W}^\varepsilon$ in (5.9) can be replaced by function $1 + \frac{\ln 2}{\ln \varepsilon}(1 - W(\frac{x}{\varepsilon}) - H^{\lambda^0}(\frac{x}{\varepsilon}))$, where H^{λ^0} is the solution of (3.3) for $\lambda = \lambda^0$, provided that λ^0 is not an eigenvalue of the local problem (3.4). On account of (3.8) and (3.9) these functions also satisfy (5.10). Its value on each B^ε is $(-\frac{\ln 2}{\ln \varepsilon}(1 - W(\frac{x-\tilde{x}_k}{\varepsilon}) - H^{\lambda^0}(\frac{x-\tilde{x}_k}{\varepsilon}))$, and considering (3.3) and (3.7) the other assertions in Proposition 2 also hold.

Remark 8. When $\alpha > 0$ and $m > 2$ we have not managed to prove similar results to those in Theorems 4 and 6 for $m = 2$, at least that u_i^ε is assumed to be 0 in $\bigcup B^\varepsilon$. Similar situations were already noticed in Ref. 4. Theorem 7 gives us a partial answer to this problem.

Theorem 7. *Let m be > 2, $\alpha = \lim_{\varepsilon\to 0} \frac{-1}{\eta \ln \varepsilon} > 0$ and $(\lambda_i^\varepsilon\,,\, u_i^\varepsilon)$ be the eigen-elements of* (2.1). *Let us suppose that there is a sequence $\varepsilon_n \to 0$ such that:*

$$\lambda_{i(\varepsilon_n)}^{\varepsilon_n} \xrightarrow{n\to\infty} \lambda^0 \quad \textit{and} \quad u_{i(\varepsilon_n)}^{\varepsilon_n} \xrightarrow{n\to\infty} u^0 \quad \textit{in } H^1(\Omega),$$

where u^0 is assumed to be nonzero. Then, (λ^0, u^0) is an eigen-element of (4.12).

Proof. As in the proof of Theorem 4 we just write $v^\varepsilon = \phi\tilde{w}^\varepsilon$ in (2.3), with $\phi \in \{u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \Gamma_\Omega\}$ and $\tilde{w}^\varepsilon$ defined in (5.12). On account of properties (a) and (b) in Proposition 3 we pass to the limit in the equation

$$\int_\Omega \nabla u^\varepsilon \cdot \nabla(\tilde{w}^\varepsilon \phi)\, dx = \lambda^\varepsilon \int_\Omega u^\varepsilon \tilde{w}^\varepsilon \phi\, dx,$$

when $\varepsilon \to 0$. We obtain,

$$\int_\Omega \nabla u^0 \cdot \nabla \phi\, dx = \lambda^0 \int_\Omega u^0 \phi\, dx,$$

i.e. the result established in the theorem. □

Remark 9. The fact that $u_{i(\varepsilon_n)}^{\varepsilon_n}$ converges to u^0 in $H^1(\Omega)$, as $\varepsilon_n \to 0$, in Theorem 7 seems not to be a restrictive condition, since each eigenfunction u^ε of (2.1) belongs to $H^{1+r}(\Omega)$ for any r with $0 < r < \frac{1}{2}$ (see Sec. 2.4 of Ref. 5). Thus, we can normalize u_i^ε in some space $H^{1+r}(\Omega)$ for a fixed $r > 0$, and then extract the sequence as the theorem states.

6. Asymptotic Behavior of the Eigenvalues for $m < 2$

In this section we study the asymptotic behavior of the eigenvalues, λ_i^ε, of (2.1), as $\varepsilon \to 0$, for $m < 2$. We use the Energy Method and typical techniques in G-convergence of operators theory (see Sec. III.9.1 of Ref. 1 and Sec. 5 of Ref. 9 for more details) to derive spectral convergence. Now, the effect of the regions B^ε where the density is higher than elsewhere, can be disregarded, as it happens in the case of a single concentrated mass (see Ref. 11). As the following theorem states, the asymptotic behavior of the eigenvalues is as if the concentrated masses did not exist, that is to say as if $m = 0$.

Theorem 8. *Let α be $\alpha = \lim_{\varepsilon\to 0} \frac{-1}{\eta \ln \varepsilon}$ and $(\lambda_i^\varepsilon, u_i^\varepsilon)$ the eigen-elements of* (2.1). *There is a sequence $\varepsilon_n \to 0$ such that for every $i \in \mathbb{N}$:*

$$\lambda_i^{\varepsilon_n} \xrightarrow{n\to\infty} \lambda_i, \tag{6.1}$$

$$u_i^{\varepsilon_n} \xrightarrow{n\to\infty} u_i \quad \textit{weakly in } H^1(\Omega), \tag{6.2}$$

where λ_i is the i-th eigenvalue of (4.11) *when $\alpha > 0$.* ((4.12) *when $\alpha = 0$ and* (4.13) *when $\alpha = +\infty$, respectively) and u_i is the corresponding eigenfunction.*

Proof. Let us first consider $\alpha \geq 0$. Considering (2.8), the orthonormality of u_i^ε in $\mathbf{V}$ and applying a diagonalization argument we can extract a sequence $\varepsilon_n \to 0$ such that $\lambda_i^{\varepsilon_n}$, $u_i^{\varepsilon_n}$ satisfy (6.1) and (6.2) for some λ_i real and $u_i \in \mathbf{V}$.

Let us consider $\phi \in \{u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \Gamma_\Omega\}$, and w^ε, the test functions defined by (5.11). We write $v^\varepsilon = \phi w^\varepsilon$ in (2.3), and we pass to the limit in the equation

$$\int_\Omega \nabla u_i^{\varepsilon_n} \cdot \nabla(w^{\varepsilon_n}\phi)\, dx = \lambda_i^{\varepsilon_n}(\frac{1}{\varepsilon_n^m} - 1)\int_{\bigcup B^\varepsilon} u_i^{\varepsilon_n} w^{\varepsilon_n}\phi\, dx + \lambda_i^{\varepsilon_n}\int_\Omega u_i^{\varepsilon_n} w^{\varepsilon_n}\phi\, dx\,,$$

when $\varepsilon_n \to 0$. Using the techniques in the proof of Theorem 4 and taking into account $m < 2$, the Poincaré inequality on each region B^ε, and properties (b), (c) and (5.13) of (d) in Proposition 2, we obtain that (λ_i, u_i) satisfy (4.14). That is to say, provided that $u_i \neq 0$, (λ_i, u_i) is an eigen-element of (4.11).

The orthonormality of u_i^ε in $\mathbf{V}$ and the Poincaré inequality on each B^ε allow us to prove that the eigenfunctions u_i are orthogonal in $L^2(\Omega)$ and $u_i \neq 0$. Therefore, the number of λ_i cannot be a finite number. We use (5.13) and a typical process of contradiction to prove that λ_i is the i-th eigenvalue of (4.11) (see Sec. III.9.1 of Ref. 1 and Sec. 6 of Ref. 9 dealing with this kind of technique).

The result for $\alpha = +\infty$ is shown using the techniques in the proof of Theorem 4 for that value of α and the above reasoning in this theorem. □

Remark 10. The case $m = 0$ has already been considered in Ref. 3 in the framework of boundary homogenization problems; Ref. 3 provides estimates for the difference between the eigen-elements $(\lambda^\varepsilon, u^\varepsilon)$ and those of the homogenized problem.

References

1. H. Attouch, **Variational Convergence for Functions and Operators** (Pitman, 1984).
2. A. Brillard, M. Lobo and E. Pérez, *Homogénéisation de frontières par epiconvergence en élasticité linéare*, *Math. Model Numer. Anal.* **24** (1990) 5–26.
3. G. A. Chechkin, *Elliptical problems with rapidly changing boundary conditions*, Doctoral dissertation, Moscow, 1992.
4. D. Cioranescu and F. Murat, *Un terme étrange venu dáilleurs*, **Nonlinear Partial Differential Equations and Their Applications**, *Collège de France Séminar, Vols. II and III*, eds. H. Brezis and J. L. Lions, *Research Notes in Math.*, Nos. 60 and 70 (Pitman, 1982) pp. 98–138 and 154–178.
5. P. Grisvard, **Singularity in Boundary Value Problems** (Springer-Verlag, 1991).
6. C. Leal and J. Sanchez-Hubert, *Perturbation of the eingenvalue of a membrane with a concentrated mass*, *Quarterly Appl. Math.*, No. 1, **XLVII** (1989) 93–103.
7. M. Lobo and E. Pérez, *Boundary homogenization of certain elliptic problems for cylindrical bodies*, *Bull. Soc. Math.* (2) **116** (1992) 399–426.
8. M. Lobo and E. Pérez, *On vibrations of a body with many concentrated masses near the boundary*, *Math. Models Methods Appl. Sci.* **3** (1993) 249–273.
9. M. Lobo and E. Pérez, *Vibrations of a body with many concentrated masses near the boundary: High frequency vibrations*, **Spectral Analysis of Complex Structures** (Hermann, 1995) pp. 85–101.

10. L. M. Milne-Thomson, **Theoretical Hydrodynamics** (Macmillan, 1968).
11. O. A. Oleinik, J. Sanchez-Hubert and G. A. Yosifian, *On vibrations of a membrane with concentrated masses*, *Bull. Sci. Math.* (*2*) **115** (1991) 1–27.
12. J. Sanchez-Hubert, *Perturbation des valeurs propres pour des systèmes avec masse concentrée. C.R. Acad. Sci. Paris Série II* **309** (1989) 505–510.
13. J. Sanchez-Hubert and E. Sanchez-Palencia, **Vibration and Coupling of Continuous Systems, Asymptotic Methods** (Springer-Verlag, 1989).