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ON THE EIGENFUNCTIONS ASSOCIATED WITH THE HIGH FREQUENCIES IN SYSTEMS WITH A CONCENTRATED MASS *

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ABSTRACT. – We consider the vibrations of a system consisting of the domain Ω of $\mathbb{R}^N$, $N = 2, 3$, that contains a small region with diameter depending on a small parameter ε. The density is of order $\mathrm{O}(\varepsilon^{-m})$ in the small region, *the concentrated mass*, and it is $\mathrm{O}(1)$ outside; m is a parameter, $m \geqslant 2$. We study the asymptotic behaviour, as $\varepsilon \to 0$, of the eigenvalues of order $\mathrm{O}(1)$, *the high frequencies* when $m > 2$, and the corresponding eigenfunctions of the associated spectral problem. We provide information on the structure of these eigenfunctions. We also check theoretical results with explicit calculations for the dimensions $N = 1$ and $N = 2$ and give correcting terms for the eigenfunctions. © Elsevier, Paris

1. Introduction

We study the high frequency vibrations of a body occupying a domain Ω of $\mathbb{R}^N$ that contains a small region of high density, the so-called *concentrated mass*. The diameter of this region, εB, is $\mathrm{O}(\varepsilon)$ while the density is $\mathrm{O}(\varepsilon^{-m})$ in εB and $\mathrm{O}(1)$ outside; m and ε are two parameters; $m \geqslant 2$ and we shall make ε go to 0. We consider the corresponding spectral problem for the Laplace operator (see (2.1)). It should be pointed out that the results and techniques in this paper are very different from those in the literature for systems with concentrated masses.

Many papers have been devoted to the study of the vibrations for systems with one single concentrated mass using different techniques: let us mention [21] for the elasticity system, [6] for rods and plates equations, and [7,8,14–16] and [18–20] for the Laplace operator. Very different cases appear according to the operator, the dimension N of the space and the value of the parameter $m > 0$. When $m > 2$, only a few of the above mentioned papers consider the high frequency vibrations, i.e., the vibrations associated with the eigenvalues λ^ε of order $\mathrm{O}(1)$; the results are obtained in terms of asymptotic expansions or convergences of spectral families: see [8] and [19–21]. See [9–11] for different results in systems with many concentrated masses.

Dealing with the low and high frequencies, when $m > 2$, a common fact which is clearly described in the literature is that two kinds of vibrations appear: *local vibrations* and *global vibrations*.

The local vibrations are those for which the corresponding eigenfunctions u^ε are significant only in a region near the concentrated mass (i.e., for $|x| = \mathrm{O}(\varepsilon)$) while they are very small at the

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distance O(1) of the concentrated mass (the order of magnitude of u^ε depends on the dimension N). The associated eigenvalues are of order $\mathrm{O}(\varepsilon^{m-2})$: *the low frequencies*.

The global vibrations affect the whole body and the corresponding eigenfunctions are small in a neighbourhood of the concentrated mass (i.e., for $|x| = \mathrm{O}(\varepsilon)$). The corresponding eigenvalues are of order O(1): *the high frequencies*.

On the basis of different approaches, the asymptotic behaviour of the low frequencies have been widely studied in [8,14–16], and [19–21]. They accumulate at the origin: once the frequencies are normalized, the values $(\lambda^\varepsilon/\varepsilon^{m-2})$ are approached by the eigenvalues the *local problem* (see (2.4)). Here, we are not concerned with these frequencies; see Lemma 2.1 for the main results that we shall use throughout this paper. To our knowledge, there is a lack of information on the behaviour of the high frequencies, as well as for the corresponding eigenfunctions, which we describe here below.

For the high frequencies, in Section VII.12 of [19], it is proved that each $\lambda \in \sigma_g$ is an accumulation point of eigenvalues λ^ε; σ_g denotes the spectrum of the Dirichlet problem (2.8). We notice that other converging sequences of eigenvalues of order O(1) could also exist. On the other hand, this result for the eigenvalues does not provide any information on the eigenfunctions, as it is obtained in terms of a very poor convergence of certain spectral families. The only information for the eigenfunctions u^ε associated with the eigenvalues $\lambda^\varepsilon \approx \lambda^0$, for $\lambda^0 \in \sigma_g$, is obtained from the matching asymptotic techniques (see [8] when $N = 2$ and Section VII.10 of [19] when $N = 3$): u^ε are approached through the eigenfunctions of the Dirichlet problem (2.8) and it seems as if they are zero in εB (see Remark 4.2).

In this paper, we prove that the high frequencies accumulate in $(0, \infty)$ (see Section 3), and we characterize the behaviour of the eigenfunctions associated with the frequencies according to whether these frequencies are asymptotically near a point of σ_g or not (see Sections 4 and 6).

In Section 4.1, we provide information about the structure of the eigenfunctions associated with the high frequencies: only those associated with eigenvalues λ^ε converging towards a point of σ_g, as $\varepsilon \to 0$, are asymptotically non-null in $L^2(\Omega)$. In addition, in Section 4.2 we prove that all the eigenfunctions have an oscillatory character in εB: in the local variable $y = x/\varepsilon$, they are approached through eigenfunctions of the local problem (2.4) associated with large frequencies.

The oscillations of the eigenfunctions associated with $\lambda^\varepsilon = \mathrm{O}(1)$ inside εB were already glimpsed in Section VII.10 of [19]. Nevertheless, it should be emphasized that it could very well occur that these eigenfunctions concentrate on a neighbourhood of the boundary εB and vanish inside εB, that is to say, some kind of *whispering gallery phenomena* would happen (see Remark 6.3). In Section 6, we prove this strongly oscillatory character on the whole εB for certain eigenfunctions. Explicit computations, when $N = 2$ and B is a circle, allow us to give a correcting term for the eigenfunctions. In particular, we improve the convergence results in Section 4 for certain sequences λ^{ε_k}, when $\varepsilon_k \to 0$.

In Section 5 we give the results for a vibrating string ($N = 1$), as we consider that they may clarify the more general results of Section 4 (see [7] for a study of the low frequencies). We observe a different behaviour for the eigenfunctions than that noted for the dimension of the space $N \geqslant 2$: when $N = 1$ all of them are strongly oscillating functions inside the concentrated mass and no whispering gallery phenomena can occur.

Finally, in Section 7, we consider the case $N = 2$ and $m = 2$: we give convergence results for the eigenelements $(\lambda^\varepsilon, u^\varepsilon)$ of (2.2).

2. Statement of the problem

Let us consider Ω an open bounded domain of $\mathbb{R}^N$, $N = 2$ or 3, with a smooth boundary $\partial\Omega$. Let B be an open bounded domain with a smooth boundary which we denote by Γ. For the sake of simplicity, we consider that both Ω and B contain the origin. Let ε be a positive small parameter that we shall make to go to 0. Let us consider εB and $\varepsilon\Gamma$ the homothetics of B and Γ respectively with ratio ε, and we assume that $\varepsilon\overline{B}$ is contained in Ω. Let m be $m > 2$.

Let us consider the spectral problem:

$$\begin{cases} -\Delta u^\varepsilon = \lambda^\varepsilon u^\varepsilon & \text{in } \Omega - \varepsilon\overline{B}, \\ -\Delta u^\varepsilon = \lambda^\varepsilon \varepsilon^{-m} u^\varepsilon & \text{in } \varepsilon B, \\ [u^\varepsilon] = \left[\dfrac{\partial u^\varepsilon}{\partial n}\right] = 0 & \text{on } \varepsilon\Gamma, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where n is the unit outward normal to $\varepsilon\Gamma$ and the brackets denote the jump across $\varepsilon\Gamma$ of the enclosed quantities.

The variational formulation of problem (2.1) is:

Find λ^ε, $u^\varepsilon \in H_0^1(\Omega)$, $u^\varepsilon \neq 0$, satisfying:

$$\int_\Omega \nabla u^\varepsilon.\nabla v \,\mathrm{d}x = \lambda^\varepsilon \left[\int_{\Omega-\varepsilon\overline{B}} u^\varepsilon v \,\mathrm{d}x + \frac{1}{\varepsilon^m}\int_{\varepsilon B} u^\varepsilon v \,\mathrm{d}x\right], \quad \forall v \in H_0^1(\Omega). \tag{2.2}$$

For each fixed $\varepsilon > 0$, (2.2) is a standard eigenvalue problem. Let us consider

$$0 < \lambda_1^\varepsilon \leqslant \lambda_2^\varepsilon \leqslant \cdots \leqslant \lambda_n^\varepsilon \leqslant \cdots \xrightarrow{n\to\infty} \infty,$$

the sequence of eigenvalues, with the classical convention of repeated eigenvalues. Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the corresponding eigenfunctions, which are assumed to be an orthonormal basis in $H_0^1(\Omega)$, i.e.,

$$\left\|\nabla u^\varepsilon\right\|_{L^2(\Omega)} = 1.$$

Let σ_ε denote the spectrum of (2.2): $\sigma_\varepsilon = \{\lambda_i^\varepsilon\}_{i=1}^\infty$.

An estimate for the eigenvalues can be obtained by using the minimax principle and the coerciveness of the form on the left-hand side of (2.2) (see [9] and [11], for example, for the technique). In fact, for each fixed $i = 1, 2, 3, \ldots$ we have:

$$\begin{aligned} C\varepsilon^{m-2}|\ln\varepsilon|^{-1} < \lambda_i^\varepsilon < C_i\varepsilon^{m-2} \quad & \text{for } N = 2, \\ C\varepsilon^{m-2} < \lambda_i^\varepsilon < C_i\varepsilon^{m-2} \quad & \text{for } N = 3, \end{aligned} \tag{2.3}$$

where C, C_i are constants independent of ε and $C_i \to \infty$ when $i \to \infty$.

Problem (2.2) has been studied by several authors using different techniques: the case when $N = 2$ is considered in [8,14,15,18,19], while the case $N = 3$ is studied in [14,16,18–20].

Regarding the low frequencies, a thorough study is performed in the above mentioned papers and we do not add anything to what has already been said. We state here the main results, obtained in these papers when $N = 2, 3$, which will be useful in Sections 4, 6 and 7.

The change of variable $y = x/\varepsilon$ leads us to the following eigenvalue *local problem*:

$$\begin{cases} -\Delta_y U = \mu U & \text{in } B, \\ -\Delta_y U = 0 & \text{in } \mathbb{R}^N - \overline{B}, \\ [U] = \left[\dfrac{\partial U}{\partial n}\right] = 0 & \text{on } \Gamma, \\ U \xrightarrow{|y|\to\infty} \begin{cases} c & \text{if } N = 2, \\ 0 & \text{if } N = 3, \end{cases} \end{cases} \tag{2.4}$$

where c is an unknown but well determined constant. This problem, posed in an unbounded domain, has the variational formulation:

Find μ and $U \in \mathcal{V}$, $U \neq 0$, satisfying:

$$\int_{\mathbb{R}^N} \nabla U.\nabla V \,\mathrm{d}y = \mu \int_B U\, V \,\mathrm{d}y, \quad \forall V \in \mathcal{V}, \tag{2.5}$$

where the space $\mathcal{V}$ is the completion of $\mathcal{D}(\mathbb{R}^N)$ for the norm

$$\|U\|_{\mathcal{V}} = \left[\int_B \left|U(y)\right|^2 \mathrm{d}y + \int_{\mathbb{R}^N} \left|\nabla_y U(y)\right|^2 \mathrm{d}y\right]^{1/2}. \tag{2.6}$$

Problem (2.5) has a discrete, non-negative spectrum (see Section IV.8 of [19], for example). Let us consider

$$0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n \leqslant \cdots \xrightarrow{n\to\infty} \infty,$$

the sequence of eigenvalues, with the classical convention of repeated eigenvalues. Let $\{U_i\}_{i=1}^{\infty}$ be the corresponding eigenfunctions suitably normalized.

Let us denote by $\|\cdot\|_1$ the norm in $H^1(B)$, equivalent to the usual norm,

$$\|U\|_1^2 = \|U\|_{H^1(B)}^2 + \langle TU, U\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}, \tag{2.7}$$

where T denotes the trace operator on Γ (see Section IV.8 of [19]). The following result is proved in [8] when $N = 2$ (Section VII.11 of [19] when $N = 3$) and in [15] (Section III.5 of [16], respectively) with other techniques of the Spectral Perturbation Theory.

LEMMA 2.1. – *Let m be $m > 2$. Let λ_i^ε be the eigenvalues of* (2.2) *and U_i^ε the corresponding eigenfunctions with norm* 1 *in $\mathcal{V}$. For fixed i, the values $\lambda_i^\varepsilon/\varepsilon^{m-2}$ converge, when $\varepsilon \to 0$, towards the eigenvalues of* (2.4) *with conservation of the multiplicity. For each sequence it is possible to extract a subsequence, still denoted by ε, such that the corresponding eigenfunctions, U_i^ε, converge towards U_i in $L^2(B)$ where U_i is an eigenfunction associated with the i-th eigenvalue of* (2.4)*, and $\{U_i\}_{i=1}^{\infty}$ form an orthonormal basis of $H^1(B)$ for the scalar product associated with the norm* (2.7).

Note, that it is an essential fact, in order to obtain the result in Lemma 2.1, to consider the microscopic variable y. In the macroscopic variable, we observe that, if $\{u_i^\varepsilon\}_{i=1}^{\infty}$ is an orthonormal basis of $H_0^1(\Omega)$, for each fixed i, $u_i^\varepsilon \xrightarrow{\varepsilon\to 0} 0$ weakly in $H_0^1(\Omega)$ (see Proposition 4.3). This suggests that other asymptotically non null eigenfunctions should exist and obviously we must look for these eigenfunctions among those associated with the high frequencies.

Dealing with the high frequencies, we outline here the main results obtained in [8,19,20]. Asymptotic expansions for the eigenvalues $\lambda^\varepsilon = \lambda + o(1)$ and for the eigenfunctions $u^\varepsilon = u + o(1)$ lead to the Dirichlet problem:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

which ignores the concentrated mass. Let

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots \xrightarrow{n\to\infty} \infty,$$

be the sequence of eigenvalues, with the classical convention of repeated eigenvalues. Let us denote σ_g the spectrum of (2.8), $\sigma_g = \{\lambda_i\}_{i=1}^{\infty}$.

The Fourier transform technique for time dependent problems provides the following result on the spectral convergence (see Section VII.12 of [19] for the proof).

LEMMA 2.2. – *For each λ_i eigenvalue of* (2.8), *there is a sequence $\lambda^\varepsilon_{i(\varepsilon)}$ of eigenvalues of* (2.2) *converging to λ_i as $\varepsilon \to 0$.*

For convenience we introduce here a result from the Spectral Perturbation Theory that will prove useful in Sections 4 and 6 (see [22] for its proof).

LEMMA 2.3. – *Let $A : H \to H$ be a linear, self-adjoint, positive and compact operator on a Hilbert space H. Let $u \in H$, with $\|u\|_H = 1$ and $\lambda,\ r > 0$ such that $\|Au - \lambda u\|_H < r$. Then, there exists an eigenvalue λ_i of A satisfying $|\lambda - \lambda_i| < r$. Moreover, for any $r^* > r$ there is $u^* \in H$ with $\|u^*\|_H = 1$ such that*

$$\left\| u - u^* \right\|_H < \frac{2r}{r^*},$$

u^* belonging to the eigenspace associated with all the eigenvalues of the operator A lying on the segment $[\lambda - r^*, \lambda + r^*]$.

3. Spectral concentration of σ_ε in $[0, \infty)$

As is well known, the classical Weyl numbers give an idea about how, for certain elliptic problems, the large frequencies are distributed; actually, depending on the dimension of the space, they can become closer and closer. A small parameter ε appearing in the problem could enlarge the field of applications of these classical results allowing the densification of the spectrum in $[0, +\infty)$ to be guessed, as $\varepsilon \to 0$ (see [5] and [12]). We prove here that this is the case for the spectrum σ_ε of (2.2).

The main result in this section is stated in Theorem 3.1. We use the method of the Fourier Transform for its proof (see [12] and Section V.13 in [19]). Note that this method is very general: all ends up as a weak convergence of the corresponding spectral families operating on certain test functions. This allows us to obtain spectral convergence results when the limit spectral family is not a constant one (see [9,17], for very different cases).

THEOREM 3.1. – *For any $\lambda^* > 0$, there is a sequence $\lambda^\varepsilon_{i(\varepsilon)}$ of eigenvalues of* (2.2) *converging towards λ^* as $\varepsilon \to 0$.*

The proof of the theorem is based on Lemma 3.1 bellow. Let us first introduce some notations which will prove to be useful for the proof. We dilate the space variable x by introducing a new variable ξ, $\xi = x/\varepsilon^{m/2}$. We denote by $\Omega_\varepsilon = \{\xi \in \mathbb{R}^N\colon\ \xi\varepsilon^{m/2} \in \Omega\}$, $\widetilde{\Omega}_\varepsilon = \{\xi \in \mathbb{R}^N\colon\ \xi\varepsilon^{m/2} \in \Omega - \varepsilon\overline{B}\}$, $\widetilde{B}_\varepsilon = \{\xi \in \mathbb{R}^N\colon\ \xi\varepsilon^{m/2} \in \varepsilon B\}$. We assume that the elements of $H_0^1(\Omega_\varepsilon)$ are extended to $\mathbb{R}^N$ with the value 0. Let us consider problem (2.2) written in the ξ variable:

$$\int_{\Omega_\varepsilon} \nabla_\xi u^\varepsilon . \nabla_\xi v \,\mathrm{d}\xi = \lambda^\varepsilon \left[\varepsilon^m \int_{\widetilde{\Omega}_\varepsilon} u^\varepsilon v \,\mathrm{d}\xi + \int_{\widetilde{B}^\varepsilon} u^\varepsilon v \,\mathrm{d}\xi \right], \quad \forall v \in H_0^1(\Omega_\varepsilon). \tag{3.1}$$

Let $\mathcal{A}^\varepsilon$ be the positive, self-adjoint, anticompact operator on $\mathcal{H}^\varepsilon$ associated with the form on the left-hand side of (3.1), where by $\mathcal{H}^\varepsilon$ we denote the space $L^2(\Omega_\varepsilon)$ with the scalar product

$$(u, v)_{\mathcal{H}^\varepsilon} = \varepsilon^m \int_{\widetilde{\Omega}_\varepsilon} uv \,\mathrm{d}\xi + \int_{\widetilde{B}_\varepsilon} uv \,\mathrm{d}\xi, \quad \forall u, v \in \mathcal{H}^\varepsilon,$$

then $\mathcal{A}^\varepsilon$ has a discrete spectrum, $\sigma_\varepsilon = \{\lambda_i^\varepsilon\}_{i=1}^\infty$. Let $\{e_i^\varepsilon\}_{i=1}^\infty$ be the corresponding eigenfunctions $e_i^\varepsilon \in H_0^1(\Omega_\varepsilon)$, which are assumed to be an orthonormal basis in $\mathcal{H}^\varepsilon$. Let $\mathcal{B}^\varepsilon$ be $\mathcal{B}^\varepsilon = \mathcal{A}^\varepsilon + I^\varepsilon$, I^ε the unitary operator in $\mathcal{H}^\varepsilon$.

Let us consider $\mathcal{A}$ the operator on $L^2(\mathbb{R}^N)$ associated with the Laplacian operator. As is known $\mathcal{A}$ is a non-negative, self-adjoint operator in $L^2(\mathbb{R}^N)$ with continuous spectrum, $\sigma(\mathcal{A}) = [0, \infty)$ (see, for example, Sections II and V of [23]). Let $\mathcal{B}$ denote the operator $\mathcal{A} + I$ where I is the unitary operator in $L^2(\mathbb{R}^N)$; in fact, $\mathcal{B}$ is the operator associated with the scalar product in $H^1(\mathbb{R}^N)$ and $\sigma(\mathcal{B}) = [1, \infty)$.

Let us consider the evolution problem associated with (3.1), in the spaces $\mathcal{H}^\varepsilon$ and $H_0^1(\Omega_\varepsilon)$, for some initial data:

$$\begin{cases} \dfrac{\mathrm{d}^2\mathbf{u}^\varepsilon}{\mathrm{d}t^2} + \mathcal{B}^\varepsilon \mathbf{u}^\varepsilon = 0, \\ \mathbf{u}^\varepsilon(0) = 0,\ \dfrac{\mathrm{d}\mathbf{u}^\varepsilon}{\mathrm{d}t}(0) = f_\varepsilon, \end{cases} \tag{3.2}$$

where the index ε denotes the restriction of the function f in $L^2(\mathbb{R}^N)$ to Ω_ε. Similarly, let the evolution problem associated with $\mathcal{B}$ be:

$$\begin{cases} \dfrac{\mathrm{d}^2\mathbf{u}^*}{\mathrm{d}t^2} + \mathcal{B}\mathbf{u}^* = 0, \\ \mathbf{u}^*(0) = 0,\ \dfrac{\mathrm{d}\mathbf{u}^*}{\mathrm{d}t}(0) = f, \end{cases} \tag{3.3}$$

in the spaces $L^2(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$.

Let us introduce a smooth function φ^ε which takes the value 1 in $|\xi| \leqslant \frac{1}{2}\varepsilon^{1-m/2}$, and 0 in $|\xi| \geqslant \varepsilon^{1-m/2}$:

$$\varphi^\varepsilon(\xi) = \varphi\left(\frac{2|\xi|}{\varepsilon^{1-m/2}}\right),$$

where $\varphi \in C^\infty(\mathbb{R})$, $\varphi(r) = 1$ if $r \leqslant 1$ and $\varphi(r) = 0$ if $r \geqslant 2$.

We use the technique in [12] and Section VII.6 of [19], with minor modifications, to prove the following result for the solutions of (3.2) and (3.3):

LEMMA 3.1. – *Let* $\mathbf{u}^\varepsilon(t)$ *be the solution of* (3.2) ($\mathbf{u}^*(t)$ *that of* (3.3)), *with values in* $H_0^1(\Omega_\varepsilon)$ ($H^1(\mathbb{R}^N)$ *respectively*), *that we assume to be extended by* 0 *to* $\mathbb{R}^N - \Omega_\varepsilon$. *We consider* $\tilde{\mathbf{u}}^\varepsilon = \mathbf{u}^\varepsilon \varphi^\varepsilon$. *Then,*

$$\begin{cases} \tilde{\mathbf{u}}^\varepsilon \xrightarrow{\varepsilon\to 0} \mathbf{u}^* & \text{in } L^\infty\big(-\infty,\infty; H^1(\mathbb{R}^N)\big) \text{ weak-}^*, \\ \dfrac{\mathrm{d}\tilde{\mathbf{u}}^\varepsilon}{\mathrm{d}t} \xrightarrow{\varepsilon\to 0} \dfrac{\mathrm{d}\mathbf{u}^*}{\mathrm{d}t} & \text{in } L^\infty\big(-\infty,\infty; L^2(\mathbb{R}^N)\big) \text{ weak-}^*. \end{cases} \tag{3.4}$$

From Lemma 3.1, we obtain the convergence of the corresponding spectral families associated with $\mathcal{A}^\varepsilon$ and $\mathcal{A}$ respectively:

$$\big(\mathcal{E}(\mathcal{A}^\varepsilon,\lambda)v_\varepsilon, w_\varepsilon\big)_{\mathcal{H}^\varepsilon} \xrightarrow{\varepsilon\to 0} \big(\mathcal{E}(\mathcal{A},\lambda)v, w\big)_{L^2(\mathbb{R}^N)} \quad \text{in } L^\infty(-\infty,\infty) \text{ weakly-}^*, \tag{3.5}$$

for any $v, w \in L^2(\mathbb{R}^N)$.

Proof of Theorem 3.1. – We show that for any $\lambda^* > 0$ and for any $\delta > 0$, there exists $j(\varepsilon)$ such that $\lambda^\varepsilon_{j(\varepsilon)} \in (\lambda^* - \delta, \lambda^* + \delta)$, for sufficiently small ε.

Taking into account the definition of $\mathcal{A}^\varepsilon$ and $\mathcal{A}$ respectively, for $\lambda \geqslant 0$, we have:

$$\big(\mathcal{E}(\mathcal{A}^\varepsilon,\lambda)v_\varepsilon, v_\varepsilon\big)_{\mathcal{H}^\varepsilon} = \Bigg(\sum_{\lambda_i^\varepsilon \leqslant \lambda} (e_i^\varepsilon, v_\varepsilon)_{\mathcal{H}^\varepsilon} e_i^\varepsilon, v_\varepsilon\Bigg)_{\mathcal{H}^\varepsilon} = \sum_{\lambda_i^\varepsilon \leqslant \lambda} (e_i^\varepsilon, v_\varepsilon)^2_{\mathcal{H}^\varepsilon}, \quad \forall v \in L^2\big(\mathbb{R}^N\big), \tag{3.6}$$

($\{e_i^\varepsilon\}_{i=1}^\infty$ being the eigenfunctions of (3.1)) and

$$\big(\mathcal{E}(\mathcal{A},\lambda)v, v\big)_{L^2(\mathbb{R}^N)} = \big\|\mathcal{E}(\mathcal{A},\lambda)v\big\|^2_{L^2(\mathbb{R}^N)} = \int\limits_{|p|\leqslant\sqrt{\lambda}} \big|\hat{v}(p)\big|^2 \,\mathrm{d}p, \quad \forall v \in L^2\big(\mathbb{R}^N\big), \tag{3.7}$$

where $\hat{v}(p)$ is the Fourier transform of $v \in L^2(\mathbb{R}^N)$.

On account of (3.5), taking derivatives in (3.6) and (3.7), we obtain:

$$\sum_{\lambda_i^\varepsilon \leqslant \lambda} (e_i^\varepsilon, v_\varepsilon)^2_{\mathcal{H}^\varepsilon} \varphi\big(\lambda_i^\varepsilon\big) \xrightarrow{\varepsilon\to 0} < \frac{1}{2}\lambda^{\frac{N-2}{2}} \int\limits_{S^{N-1}} \big|\hat{v}\big(\sqrt{\lambda}\eta\big)\big|^2 \,\mathrm{d}\eta, \quad \varphi >_{\mathcal{S}'(\mathbb{R})\times\mathcal{S}(\mathbb{R})}, \ \forall \varphi \in \mathcal{S}(\mathbb{R}), \tag{3.8}$$

where S^{N-1} is the unit sphere in $\mathbb{R}^N$. Then, taking v and $\varphi \in \mathcal{D}(\lambda^* - \delta, \lambda^* + \delta)$, in such a way that the right hand side of (3.8) be different from zero, we deduce that, for sufficiently small ε, there are $\lambda^\varepsilon_{j(\varepsilon)}$ such that $\varphi(\lambda^\varepsilon_{j(\varepsilon)}) \neq 0$, so that $\lambda^\varepsilon_{j(\varepsilon)} \in (\lambda^* - \delta, \lambda^* + \delta)$, and Theorem 3.1 is proved. □

4. On the structure of the eigenfunctions associated with the high frequencies

Convergence results for high frequencies are given in Lemma 2.2 and Theorem 3.1 in terms of the convergence of certain spectral families. However, no information about the structure of the corresponding eigenfunctions seems to be obtained from these results. Two different operators are obtained: the first one (associated with the Dirichlet problem (2.8)) has a pure point spectrum, σ_g, while the second one has a continuous spectrum, $[0,\infty)$ (see [12] for an analogous situation). As $\sigma_g \subset (0,\infty)$, the eigenfunctions associated with eigenvalues λ^ε asymptotically near the points of σ_g should have a different behaviour from those associated with the rest of the eigenvalues.

The aim of Section 4.1 is to differentiate this behaviour. In Section 4.2 we give information about the structure of the eigenfunctions inside εB.

4.1. Global behaviour

The main result in this section is stated in the following theorem:

THEOREM 4.1. – *Let λ be any positive real number. Let I_{δ^ε} denote an interval $[\lambda-\delta^\varepsilon, \lambda+\delta^\varepsilon]$ having eigenvalues $\lambda^\varepsilon_{i(\varepsilon)}$ of* (2.2), *with $\delta^\varepsilon \to 0$ as $\varepsilon \to 0$. Then, $\lambda \in \sigma_g$ if and only if there are $\{\delta^\varepsilon\}_\varepsilon$ and $\{\tilde{u}^\varepsilon\}_\varepsilon$, each $\tilde{u}^\varepsilon$ with norm 1 in $H^1_0(\Omega)$, belonging to the eigenspace associated with all the eigenvalues in I_{δ^ε} and such that $\|\tilde{u}^\varepsilon\|_{L^2(\Omega)} > a > 0$ for some constant a independent of ε. Moreover, if $\lambda \in \sigma_g$ and u^* is an eigenfunction associated with λ, u^* with norm 1 in $H^1_0(\Omega)$, then the sequence $\tilde{u}^\varepsilon$ converge towards u^* in $H^1_0(\Omega)$.*

The proof of the Theorem is a consequence of Propositions 4.1 and 4.2 below.

PROPOSITION 4.1. – *Let $\lambda \in \sigma_g$, and u^* be an associated eigenfunction such that $\|\nabla u^*\|_{L^2(\Omega)} = 1$. Then, there is a sequence $\{d^\varepsilon\}_\varepsilon$, $d^\varepsilon \to 0$ as $\varepsilon \to 0$, such that $[\lambda - d^\varepsilon, \lambda + d^\varepsilon]$ contains eigenvalues of* (2.2): *$\lambda^\varepsilon_{i(\varepsilon)}, \lambda^\varepsilon_{i(\varepsilon)+1}, \ldots, \lambda^\varepsilon_{i(\varepsilon)+\mathrm{k}(\varepsilon)}$. Moreover, there is $\tilde{u}^\varepsilon_{i(\varepsilon)} \in H^1_0(\Omega)$, with $\|\nabla \tilde{u}^\varepsilon_{i(\varepsilon)}\|_{L^2(\Omega)} = 1$, $\tilde{u}^\varepsilon_{i(\varepsilon)}$ belonging to the eigenspace associated with the eigenvalues $\lambda^\varepsilon_{i(\varepsilon)}$ in $[\lambda - d^\varepsilon, \lambda + d^\varepsilon]$, such that*

$$\left\|\tilde{u}^\varepsilon_{i(\varepsilon)}\right\|_{L^2(\Omega)} \geqslant a > 0,$$

for some constant $a > 0$, and $\tilde{u}^\varepsilon_{i(\varepsilon)}$ converging towards u^* in $H^1_0(\Omega)$ as $\varepsilon \to 0$.

Proof. – Let A^ε be the positive, compact and symmetric operator on $H^1_0(\Omega)$ defined by:

$$\left\langle A^\varepsilon u, v\right\rangle_{H^1_0(\Omega)} = \int\limits_{\Omega-\varepsilon\overline{B}} uv\,\mathrm{d}x + \frac{1}{\varepsilon^m}\int\limits_{\varepsilon B} uv\,\mathrm{d}x, \quad \forall u, v \in H^1_0(\Omega). \tag{4.1}$$

The eigenvalues of A^ε are $1/\lambda^\varepsilon$, λ^ε being the eigenvalues of (2.2).

Let $u^* \in H^1_0(\Omega)$ be as the proposition states. For sufficiently small ε, we construct $v^\varepsilon \in H^1_0(\Omega)$ such that $v^\varepsilon = 0$ in εB and

$$\left\|u^* - v^\varepsilon\right\|_{H^1_0(\Omega)} \leqslant C\rho^\varepsilon \tag{4.2}$$

where C always denote some constant independent of ε and

$$\rho^\varepsilon = |\ln\varepsilon|^{-1/2} \quad \text{when } N=2 \quad \text{and} \quad \rho^\varepsilon = \varepsilon^{1/2} \quad \text{when } N=3. \tag{4.3}$$

In order to prove this result, it suffices to take:

$$v^\varepsilon(x) = \begin{cases} u^*(x) & \text{if } |x| > \sqrt{\varepsilon}, \\ u^*(x)\dfrac{\ln(|x|/\varepsilon)}{\ln(1/\sqrt{\varepsilon})} & \text{if } \varepsilon \leqslant |x| \leqslant \sqrt{\varepsilon}, \quad \text{if } N=2, \\ 0 & \text{if } |x| < \varepsilon, \end{cases} \tag{4.4}$$

and $v^\varepsilon = u^*\psi^\varepsilon$ if $N = 3$, where $\psi^\varepsilon(x) = \psi(|x|/\varepsilon)$ with $\psi \in C^\infty(\mathbb{R})$, $\psi(r) = 1$ if $r > 2$ and $\psi(r) = 0$ if $r < 1$.

Taking into account the definition of A^ε, the fact that v^ε vanishes in εB and that (λ, u^*) is an eigenelement of (2.8), we have:

$$\left\langle A^\varepsilon v^\varepsilon - \frac{1}{\lambda} v^\varepsilon, v \right\rangle_{H_0^1(\Omega)} = \int_\Omega \big(v^\varepsilon - u^*\big) v \, \mathrm{d}x - \frac{1}{\lambda} \int_\Omega \nabla\big(v^\varepsilon - u^*\big).\nabla v \, \mathrm{d}x, \quad \forall v \in H_0^1(\Omega).$$

We apply Schwarz and Poincaré inequalities to obtain:

$$\left| \left\langle A^\varepsilon v^\varepsilon - \frac{1}{\lambda} v^\varepsilon, v \right\rangle_{H_0^1(\Omega)} \right| \leqslant C \left\| v^\varepsilon - u^* \right\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

Hence, from (4.2), for ρ^ε defined in (4.3), it is evident that v^ε converges towards u^* in $H_0^1(\Omega)$ as $\varepsilon \to 0$, and, taking $\tilde{v}^\varepsilon = v^\varepsilon / \|v^\varepsilon\|_{H_0^1(\Omega)}$, for sufficiently small ε, we have:

$$\left\| A^\varepsilon \tilde{v}^\varepsilon - \frac{1}{\lambda} \tilde{v}^\varepsilon \right\|_{H_0^1(\Omega)} \leqslant C\rho^\varepsilon. \tag{4.5}$$

We apply Lemma 2.3 with $A = A^\varepsilon$, $H = H_0^1(\Omega)$, $u = \tilde{v}^\varepsilon$, $r = C\rho^\varepsilon$ and, for example, $r^* = 2\rho_\varepsilon^{1/2}$. We deduce that there are eigenvalues, $\{(\lambda_{i(\varepsilon)+j}^\varepsilon)^{-1}\}_{j=1}^{\mathrm{k}(\varepsilon)}$, of A^ε in $I_\varepsilon = [(1/\lambda) - 2\rho_\varepsilon^{1/2}, (1/\lambda) + 2\rho_\varepsilon^{1/2}]$. Moreover, for each ε, there is $\tilde{u}_{i(\varepsilon)}^\varepsilon \in H_0^1(\Omega)$, with $\|\tilde{u}_{i(\varepsilon)}^\varepsilon\|_{H_0^1(\Omega)} = 1$, $\tilde{u}_{i(\varepsilon)}^\varepsilon$ belonging to the eigenspace associated with all of the eigenvalues in I_ε, such that

$$\left\| \tilde{u}_{i(\varepsilon)}^\varepsilon - \tilde{v}^\varepsilon \right\|_{H_0^1(\Omega)} \leqslant C\rho_\varepsilon^{1/2}. \tag{4.6}$$

On account of $\tilde{v}^\varepsilon \to u^*$ in $H_0^1(\Omega)$ as $\varepsilon \to 0$, (4.6) leads us to assert that $\tilde{u}_{i(\varepsilon)}^\varepsilon \to u^*$ in $H_0^1(\Omega)$ as $\varepsilon \to 0$. The fact that (λ, u^*) is an eigenelement of (2.8) reads:

$$\left\| \tilde{u}_{i(\varepsilon)}^\varepsilon \right\|_{L^2(\Omega)} \to \frac{1}{\sqrt{\lambda}} \neq 0,$$

consequently, there exists a constant $a > 0$, independent of ε, such that $\|\tilde{u}_{i(\varepsilon)}^\varepsilon\|_{L^2(\Omega)} \geqslant a > 0$, for sufficiently small ε. Then, we have proved that the statements in Proposition 4.1 hold for $d^\varepsilon = \mathrm{O}(\rho_\varepsilon^{1/2})$. □

PROPOSITION 4.2. – *Let us consider $\lambda > 0$, let $\{\delta^\varepsilon\}_\varepsilon$ be any sequence such that $\delta^\varepsilon \to 0$ as $\varepsilon \to 0$, let $\{\lambda_{i(\varepsilon)}^\varepsilon, \lambda_{i(\varepsilon)+1}^\varepsilon, \ldots, \lambda_{i(\varepsilon)+\mathrm{k}(\varepsilon)}^\varepsilon\}$ be all the eigenvalues of* (2.2) *in $[\lambda - \delta^\varepsilon, \lambda + \delta^\varepsilon]$, and $\tilde{u}^\varepsilon$ any function in the eigenspace $[u_{i(\varepsilon)}^\varepsilon, u_{i(\varepsilon)+1}^\varepsilon, \ldots, u_{i(\varepsilon)+\mathrm{k}(\varepsilon)}^\varepsilon]$ with $\|\nabla \tilde{u}^\varepsilon\|_{L^2(\Omega)} = 1$.*

(a) *If there is some converging subsequence $\{\tilde{u}^{\varepsilon_k}\}_k$ such that $\|\tilde{u}^{\varepsilon_k}\|_{L^2(\Omega)} > a > 0$, for some constant a independent of ε_k, then (λ, u^*) is an eigenelement of* (2.8)*, where u^* is the limit of $\tilde{u}^{\varepsilon_k}$ in $L^2(\Omega)$ as $\varepsilon_k \to 0$.*

(b) *If $\lambda \notin \sigma_g$, then*

$$\left\| \tilde{u}^\varepsilon \right\|_{L^2(\Omega)} \to 0, \quad \text{as } \varepsilon \to 0.$$

Proof. – We prove assertion (b) by contradiction. Let us suppose that λ is not an eigenvalue of (2.8), and the sequence $\|\tilde{u}^\varepsilon\|_{L^2(\Omega)}$ does not converge to zero as $\varepsilon \to 0$. Because of the boundedness of $\tilde{u}^\varepsilon$ in $H_0^1(\Omega)$ we can assert that there is a subsequence $\{\tilde{u}^{\varepsilon_k}\}_k$ converging weakly in $H_0^1(\Omega)$ towards some function u^*, as $\varepsilon \to 0$, with $u^* \neq 0$.

Let us consider $\tilde{u}^\varepsilon = \sum_{j=1}^{\mathrm{k}(\varepsilon)} \beta_j^\varepsilon u_{i(\varepsilon)+j}^\varepsilon$, for certain β_j^ε; on account of relation (2.2) for each eigenelement $(\lambda_{i(\varepsilon)+j}^\varepsilon, u_{i(\varepsilon)+j}^\varepsilon)$, $j = 1, \ldots, \mathrm{k}(\varepsilon)$, we obtain $\forall v \in H_0^1(\Omega)$:

$$\int_\Omega \nabla\tilde{u}^\varepsilon.\nabla v\,\mathrm{d}x = \lambda \int_{\Omega-\varepsilon\overline{B}} \tilde{u}^\varepsilon v\,\mathrm{d}x + \frac{\lambda}{\varepsilon^m}\int_{\varepsilon B} \tilde{u}^\varepsilon v\,\mathrm{d}x + \sum_{j=1}^{\mathrm{k}(\varepsilon)} \bigl(\lambda_{i(\varepsilon)+j}^\varepsilon - \lambda\bigr)\beta_j^\varepsilon \int_{\Omega-\varepsilon\overline{B}} u_{i(\varepsilon)+j}^\varepsilon v\,\mathrm{d}x$$
$$+ \frac{1}{\varepsilon^m}\sum_{j=1}^{\mathrm{k}(\varepsilon)} \bigl(\lambda_{i(\varepsilon)+j}^\varepsilon - \lambda\bigr)\beta_j^\varepsilon \int_{\varepsilon B} u_{i(\varepsilon)+j}^\varepsilon v\,\mathrm{d}x.$$

Taking $v \in H_0^1(\Omega)$ vanishing in a neighbourhood of the origin, we obtain, for sufficiently small ε, the relation

$$\int_\Omega \nabla\tilde{u}^\varepsilon.\nabla v\,\mathrm{d}x = \lambda \int_\Omega \tilde{u}^\varepsilon v\,\mathrm{d}x + \sum_{j=1}^{\mathrm{k}(\varepsilon)} \bigl(\lambda_{i(\varepsilon)+j}^\varepsilon - \lambda\bigr)\beta_j^\varepsilon \int_\Omega u_{i(\varepsilon)+j}^\varepsilon v\,\mathrm{d}x. \tag{4.7}$$

Then, on account of $|\lambda_{i(\varepsilon_k)+j}^{\varepsilon_k} - \lambda| \leqslant \delta^{\varepsilon_k}$ and $\sum_{j=1}^{\mathrm{k}(\varepsilon)}(\beta_j^\varepsilon)^2 = 1$, we take limits in (4.7) when $\varepsilon_k \to 0$, to obtain that u^* satisfies:

$$\int_\Omega \nabla u^*.\nabla v\,\mathrm{d}x = \lambda \int_\Omega u^* v\,\mathrm{d}x.$$

Now, as the set of the functions in $H_0^1(\Omega)$ vanishing in a neighbourhood of the origin is dense in $H_0^1(\Omega)$, and $u^* \neq 0$, we obtain that λ is an eigenvalue of (2.8) which contradicts the hypothesis. Therefore, result (b) holds. It is evident that the proof of assertion (a) is contained in the previous demonstration. □

COROLLARY 4.1. – *Let λ be a positive real number, $\lambda \notin \sigma_g$, and let $\{\lambda_{i(\varepsilon)}^\varepsilon\}_\varepsilon$ be any sequence of eigenvalues of* (2.2) *converging towards λ as $\varepsilon \to 0$. Then, the corresponding eigenfunctions $u_{i(\varepsilon)}^\varepsilon$ converge towards* 0 *in $L^2(\Omega)$, as $\varepsilon \to 0$.*

Remark 4.1. – In the proof of Proposition 4.1 we have obtained that $d^\varepsilon = \mathrm{O}(\rho_\varepsilon^{1/2})$. In fact, it can be $d^\varepsilon = \mathrm{O}((\rho^\varepsilon)^{1-\beta})$ for $0 < \beta < 1$, and $\|\tilde{u}_{i(\varepsilon)}^\varepsilon - \tilde{v}^\varepsilon\|_{H_0^1(\Omega)} \leqslant C\rho_\varepsilon^\beta$ instead of (4.6). If there is only one eigenvalue, $\lambda_{i(\varepsilon)}^\varepsilon$, in $[\lambda - d^\varepsilon, \lambda + d^\varepsilon]$, with multiplicity 1, then, the function $\tilde{u}_{i(\varepsilon)}^\varepsilon$ is the corresponding eigenfunction. Moreover, in this case, Proposition 4.1 and Corollary 4.1 characterize the eigenfunctions associated with eigenvalues $\lambda^\varepsilon \approx \lambda$: if $\lambda \in \sigma_g$, then $u_{i(\varepsilon)}^\varepsilon \to u^*$ in $H_0^1(\Omega)$, where u^* is the eigenfunction associated with λ with norm 1 in $H_0^1(\Omega)$; if $\lambda \notin \sigma_g$, then $u_{i(\varepsilon)}^\varepsilon \to 0$ in $L^2(\Omega)$. That is to say, the eigenfunctions are asymptotically different from zero if and only if the corresponding eigenvalues converge towards an eigenvalue of (2.8). □

In addition to the result in Corollary 4.1, in the following proposition, we prove that the only converging sequences $\lambda^\varepsilon/\varepsilon^\alpha$ giving rise to global vibrations of the whole body are those associated with the eigenvalues of order 1 that converge towards an eigenvalue of the Dirichlet problem (2.8).

PROPOSITION 4.3. – *Let us assume that $\{\lambda_{i(\varepsilon)}^\varepsilon\}_\varepsilon$ is any sequence of eigenvalues of* (2.2) *such that $\lambda_{i(\varepsilon)}^\varepsilon/\varepsilon^\alpha$ converge towards some $\lambda \neq 0$, as $\varepsilon \to 0$, for some α, $0 < \alpha \leqslant m - 2$ or $\alpha < 0$. Then, the corresponding eigenfunctions $u_{i(\varepsilon)}^\varepsilon$ converge towards* 0 *in $L^2(\Omega)$, as $\varepsilon \to 0$.*

Proof. – We suppose that there is a sequence $\lambda^\varepsilon_{i(\varepsilon)}$ such that $\lambda^\varepsilon_{i(\varepsilon)}/\varepsilon^\alpha \xrightarrow{\varepsilon\to 0} \lambda^*$ and the corresponding eigenfunctions satisfy $u^\varepsilon_{i(\varepsilon)} \xrightarrow{\varepsilon\to 0} u^* \neq 0$ weakly in $H^1_0(\Omega)$. Then, on account that the set of the functions of $H^1_0(\Omega)$ vanishing in a neighbourhood of the origin is dense in $H^1_0(\Omega)$, we pass to the limit in the equation (2.2) and we obtain that $u^* = 0$, which is a contradiction. □

Remark 4.2. – We observe that the eigenfunctions u^ε, with norm 1 in $H^1_0(\Omega)$, associated with eigenvalues λ^ε converging towards $\lambda \neq 0$ satisfy $\|u^\varepsilon\|^2_{L^2(\varepsilon B)} \leqslant C\varepsilon^m$, i.e., $\|u^\varepsilon\|_{L^2(\varepsilon B)} \to 0$ when $\varepsilon \to 0$. This result agrees with the result obtained in [8] and Section VII.10 of [19] using the method of matched asymptotic expansions. In particular, for $N = 2$, B the unit circle and (λ, u^*) an eigenelement of (2.8) (see [8]):

$$u^\varepsilon \approx u^*(x) + \left(\frac{-1}{\ln\varepsilon} W\left(\frac{x}{\varepsilon}\right) - 1\right) u^*(x),$$

where

$$\left(\frac{-1}{\ln\varepsilon} W\left(\frac{x}{\varepsilon}\right) - 1\right) u^*(x)$$

takes the value $-u^*(x)$ in εB and $(-1/\ln\varepsilon)\ln|x/\varepsilon| u^*(x)$ outside. In addition, estimate (4.6) in Proposition 4.1 also holds for

$$v^\varepsilon = u^*(x) + \left(\frac{-1}{\ln\varepsilon} W\left(\frac{x}{\varepsilon}\right) - 1\right) u^*(x)$$

which proves that

$$\left(\frac{-1}{\ln\varepsilon} W\left(\frac{x}{\varepsilon}\right) - 1\right) u^*(x)$$

is a correcting term for u^ε. Nevertheless, as pointed out in [19], in εB the wavelength is very short and eigenfunctions are expected to have a strongly oscillatory behaviour. Obviously, it is necessary to introduce the microscopic variable y in order to see this behaviour. □

4.2. Local behaviour

We are interested in the behaviour of the eigenfunctions associated with the eigenvalues of order O(1) inside the concentrated mass, hence, we perform the change of variable $y = x/\varepsilon$. Througout the rest of the section we assume that the eigenfunctions $\{U^\varepsilon_i\}_{i=1}^\infty$, are normalized in the local variable y, that is to say:

$$\left\|U^\varepsilon_i(y)\right\|_{\mathcal{V}} = \left\|U^\varepsilon_i(y)\right\|^2_{L^2(B)} + \left\|\nabla_y U^\varepsilon_i(y)\right\|^2_{L^2(\mathbb{R}^N)} = 1, \tag{4.8}$$

$\mathcal{V}$ being the space defined in (2.6). We prove that the eigenfunctions associated with the frequencies $\lambda^\varepsilon = \mathrm{O}(1)$ have a strongly oscillatory character in B as stated in the following Theorem (compare with Theorems 6.1 and 6.2).

THEOREM 4.2. – *Let λ a positive real number, and let $\lambda^\varepsilon_{i(\varepsilon)}$ be a sequence of eigenvalues of* (2.2) *converging towards λ, as $\varepsilon \to 0$. Then, the corresponding eigenfunctions $U^\varepsilon_{i(\varepsilon)}$, satisfying* (4.8)*, verify*:

$$\text{(4.9)} \qquad \textit{either} \quad U^{\varepsilon}_{i(\varepsilon)}(y) = \sum_{j=p(\varepsilon)}^{q(\varepsilon)} \alpha^{\varepsilon}_j U_j + \mathrm{o}_{\varepsilon}(1) \quad \textit{in } H^1(B),$$

$$\textit{or} \quad \big\| U^{\varepsilon}_{i(\varepsilon)} \big\|_{H^1(B)} \xrightarrow{\varepsilon\to 0} 0,$$

where α^{ε}_j *are the Fourier coefficients of the expansion of* $U^{\varepsilon}_{i(\varepsilon)}|_B$ *in Fourier series of the eigenfunctions* $\{U_j\}_{j=1}^{\infty}$ *of* (2.4)*, and,* $p(\varepsilon), q(\varepsilon)$ *are two functions converging to* ∞ *as* $\varepsilon \to 0$.

Proof. – Let p be any fixed integer, $p \geqslant 1$; let $\{\lambda^{\varepsilon}_j\}_{j=1}^{p}$ be the first p eigenvalues of problem (2.2), and $\{U^{\varepsilon}_j\}_{j=1}^{p}$ the corresponding eigenfunctions. Lemma 2.1 allows us to assert that, for each sequence there is a subsequence still denoted by ε such that for $j = 1, 2, \ldots, p$

$$\text{(4.10)} \qquad U^{\varepsilon}_j = U_j + r^{\varepsilon}_j \quad \text{in } L^2(B),$$

where r^{ε}_j converges to 0 in $L^2(B)$, as $\varepsilon \to 0$.

Because of estimates (2.3), for $j = 1, \ldots, p$, λ^{ε}_j is different from $\lambda^{\varepsilon}_{i(\varepsilon)}$, and the orthogonality of the eigenfunctions U^{ε}_j in $H^1_0(\Omega)$ leads us to prove that

$$\lim_{\varepsilon\to 0} \int_B U^{\varepsilon}_j U^{\varepsilon}_{i(\varepsilon)}\, \mathrm{d}y = 0.$$

Besides, $U^{\varepsilon}_{i(\varepsilon)}|_B$ can be expanded in Fourier series of eigenfunctions of (2.4), that is to say,

$$\text{(4.11)} \qquad U^{\varepsilon}_{i(\varepsilon)}\big|_B = \sum_{j=1}^{\infty} \alpha^{\varepsilon}_j U_j \quad \text{in } H^1(B), \ \text{with } \alpha^{\varepsilon}_j = \big(U^{\varepsilon}_{i(\varepsilon)}\big|_B, U_j \big)_1,$$

where $(U, V)_1$ is the scalar product associated with the norm (2.7); on account of $\|U^{\varepsilon}_{i(\varepsilon)}\|_{\mathcal{V}} = 1$, the α^{ε}_j are bounded by a constant independent of ε.

Thus, considering (4.10), we can write

$$0 = \lim_{\varepsilon\to 0} \sum_{j=1}^{p} \alpha^{\varepsilon}_j \big(U^{\varepsilon}_j, U^{\varepsilon}_{i(\varepsilon)} \big)_{L^2(B)} = \lim_{\varepsilon\to 0} \sum_{j=1}^{p} \alpha^{\varepsilon}_j \big(U_j, U^{\varepsilon}_{i(\varepsilon)} \big)_{L^2(B)} + \lim_{\varepsilon\to 0} \sum_{j=1}^{p} \alpha^{\varepsilon}_j \big(r^{\varepsilon}_j, U^{\varepsilon}_{i(\varepsilon)} \big)_{L^2(B)}.$$

Using (4.11) and the orthogonality of the eigenfunctions U_j in $L^2(B)$, we have

$$\text{(4.12)} \qquad \lim_{\varepsilon\to 0} \sum_{j=1}^{p} \big|\alpha^{\varepsilon}_j\big|^2 \|U_j\|^2_{L^2(B)} = -\lim_{\varepsilon\to 0} \sum_{j=1}^{p} \alpha^{\varepsilon}_j \big(r^{\varepsilon}_j, U^{\varepsilon}_{i(\varepsilon)} \big)_{L^2(B)}.$$

Hence, the convergence

$$\sum_{j=1}^{p} \big|\alpha^{\varepsilon}_j\big|^2 \xrightarrow{\varepsilon\to 0} 0,$$

holds for each fixed p. Then, using a classic argument of diagonalization (see Section I.2 of [2]), we can assert that there is a sequence $p(\varepsilon)$ converging to ∞ as $\varepsilon \to 0$ such that

$$\left\| U^{\varepsilon}_{i(\varepsilon)}\big|_B - \sum_{j=p(\varepsilon)}^{\infty} \alpha^{\varepsilon}_j U_j \right\|_{H^1(B)} \xrightarrow{\varepsilon\to 0} 0.$$

Therefore, the Theorem is proved. □

Remark 4.3. – We note that the result of Theorem 4.2 also holds in the case where the eigenfunctions $U^{\varepsilon}_{i(\varepsilon)}$ are replaced by the functions $\tilde{u}^{\varepsilon}$ arising in Propositions 4.1 and 4.2, with minor modifications.

Remark 4.4. – Taking limits in (2.2), once we have performed the change of variable $x = y\varepsilon$ and multiplied the equation by ε^{m-2}, we easily prove that the eigenfunctions $U^{\varepsilon}_{i(\varepsilon)}$ arising in Theorem 4.2 converge towards 0 in $L^2(B)$, as $\varepsilon \to 0$. Thus, Theorem 4.2 provides information about the gradient of the eigenfunctions.

5. Results for dimension $N = 1$

Let us consider the eigenvalue problem (2.2) in relation to the vibrations of a string placed in $(-1, 1)$ with the concentrated mass in $(-\varepsilon, \varepsilon)$:

$$\begin{cases} -\dfrac{\mathrm{d}^2 u^{\varepsilon}}{\mathrm{d}x^2} = \lambda^{\varepsilon} u^{\varepsilon} & \text{in } (-1, -\varepsilon) \cup (\varepsilon, 1), \\ -\dfrac{\mathrm{d}^2 u^{\varepsilon}}{\mathrm{d}x^2} = \dfrac{\lambda^{\varepsilon}}{\varepsilon^m} u^{\varepsilon} & \text{in } (-\varepsilon, \varepsilon), \\ u^{\varepsilon}(-\varepsilon^-) = u^{\varepsilon}(-\varepsilon^+),\ u^{\varepsilon}(\varepsilon^-) = u^{\varepsilon}(\varepsilon^+), \\ \dfrac{\mathrm{d}u^{\varepsilon}}{\mathrm{d}x}(-\varepsilon^-) = \dfrac{\mathrm{d}u^{\varepsilon}}{\mathrm{d}x}(-\varepsilon^+),\ \dfrac{\mathrm{d}u^{\varepsilon}}{\mathrm{d}x}(\varepsilon^-) = \dfrac{\mathrm{d}u^{\varepsilon}}{\mathrm{d}x}(\varepsilon^+), \\ u^{\varepsilon}(-1) = 0,\ u^{\varepsilon}(1) = 0. \end{cases} \tag{5.1}$$

This problem was considered in [4] and Section VII.13 of [19] when $m = 1$ and in [7] and [14] when $m > 0$. Here, we study the asymptotic behaviour of the large eigenvalues, $\lambda^{\varepsilon} = \mathrm{O}(1)$, when $m > 2$, a case which has not been considered in any of the above mentioned papers.

In the present case, results in Section 4 can be improved because explicit computations on the eigenvalues and eigenfunctions of (5.1) can be performed and the multiplicity of the eigenvalues is equal to 1. In fact, all the results in this section can be obtained by means of explicit calculations, without using the technique outlined in the previous sections. We only present the main formulas in order to illustrate the previous results.

With the exception of the eigenvalues λ^{ε}, such that $\cos(\sqrt{\lambda^{\varepsilon}}\varepsilon^{1-m/2}) = 0$ (see Remark 5.2) simple calculations show us that the eigenvalues λ^{ε} of (5.1) are the λ roots of the equation

$$\tan\left(\sqrt{\lambda}\, \varepsilon^{1-m/2}\right) = \varepsilon^{m/2} \cot\left(\sqrt{\lambda}(1-\varepsilon)\right), \tag{5.2}$$

or of the equation

$$\tan\left(\sqrt{\lambda}\, \varepsilon^{1-m/2}\right) = -\varepsilon^{-m/2} \tan\left(\sqrt{\lambda}(1-\varepsilon)\right). \tag{5.3}$$

In addition, the eigenfunctions associated with the eigenvalues λ^ε, root of the equation (5.2), are the even functions

$$u^\varepsilon(x)=\begin{cases} A^\varepsilon\alpha^\varepsilon\sin\left(\sqrt{\lambda^\varepsilon}(1+x)\right) & \text{if } x\in(-1,-\varepsilon),\\ A^\varepsilon\cos\left(\sqrt{\lambda^\varepsilon}\varepsilon^{-m/2}x\right) & \text{if } x\in(-\varepsilon,\varepsilon),\\ A^\varepsilon\alpha^\varepsilon\sin\left(\sqrt{\lambda^\varepsilon}(1-x)\right) & \text{if } x\in(\varepsilon,1),\end{cases}\tag{5.4}$$

and the eigenfunctions associated with the eigenvalues λ^ε, root of the equation (5.3), are the odd functions

$$u^\varepsilon(x)=\begin{cases} B^\varepsilon\beta^\varepsilon\sin\left(\sqrt{\lambda^\varepsilon}(1+x)\right) & \text{if } x\in(-1,-\varepsilon),\\ B^\varepsilon\sin\left(\sqrt{\lambda^\varepsilon}\varepsilon^{-m/2}x\right) & \text{if } x\in(-\varepsilon,\varepsilon),\\ -B^\varepsilon\beta^\varepsilon\sin\left(\sqrt{\lambda^\varepsilon}(1-x)\right) & \text{if } x\in(\varepsilon,1),\end{cases}\tag{5.5}$$

where α^ε, β^ε, A^ε and B^ε are constants such that

$$\int_{-1}^{1}\left|\frac{\mathrm{d}u^\varepsilon}{\mathrm{d}x}(x)\right|^2\mathrm{d}x=1.$$

We observe that for $\lambda^\varepsilon=\mathrm{O}(1)$, the eigenfunctions (5.4) and (5.5) are strongly oscillating functions in $(-\varepsilon,\varepsilon)$. Moreover, if $\{\lambda^\varepsilon_{i(\varepsilon)}\}$ is a sequence converging towards a positive number λ, then the coefficients $A^\varepsilon_{i(\varepsilon)}$ and $B^\varepsilon_{i(\varepsilon)}$ converge to 0 as $\varepsilon\to 0$. Thus, the eigenfunctions associated with the high frequencies have a small amplitude in the concentrated mass.

The corresponding local problem is posed now in $(-1,1)$,

$$\begin{cases} -\dfrac{\mathrm{d}^2U}{\mathrm{d}y^2}=\mu U \quad \text{in } (-1,1),\\ \dfrac{\mathrm{d}U}{\mathrm{d}y}(-1)=\dfrac{\mathrm{d}U}{\mathrm{d}y}(1)=0.\end{cases}\tag{5.6}$$

The eigenvalues of (5.6) are $\mu_{2k}=(k\pi)^2$, and $\mu_{2k+1}=((2k+1)\pi/2)^2$ where $k=0,1,2,\ldots,$ and the corresponding eigenfunctions (up to a constant) are:

$$U_{2k}(y)=\cos(k\pi y),\quad U_{2k+1}(y)=\sin\left(\frac{(2k+1)\pi}{2}y\right)\quad \text{respectively, for } y\in(-1,1).$$

The analogous problem to the Dirichlet problem (2.8) is:

$$\begin{cases} -\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2}=\lambda u \quad \text{in } (-1,0)\cup(0,1),\\ u(-1)=u(0)=u(1)=0.\end{cases}\tag{5.7}$$

We observe that, in this case, this limit problem is affected by the concentrated mass, as the eigenfunctions satisfy $u(0)=0$. The eigenvalues of (5.7) are $\lambda_k=(k\pi)^2$, $k=1,2,\ldots,$ with multiplicity 2, and the corresponding eigenfunctions (up to a constant), an odd and an even one respectively, are

$$u_{k,1}(x)=\sin(k\pi x)\quad \text{if } x\in(-1,0)\cup(0,1),$$

$$u_{k,2}(x)=\begin{cases}-\sin(k\pi x) & \text{if } x\in(-1,0),\\ \sin(k\pi x) & \text{if } x\in(0,1).\end{cases}$$

Theorem 3.1 allows us to assert that for each $\lambda>0$ there is a sequence $\lambda^{\varepsilon}_{i(\varepsilon)}$ of eigenvalues of (5.1) converging towards λ when $\varepsilon\to 0$.

By means of explicit calculations and using an argument of diagonalization (see Section I.2 in [2]), results for the eigenfunctions in Theorems 4.1 and 4.2 can now be stated in the following way: for any sequence $\lambda^{\varepsilon}_{i(\varepsilon)}\to\lambda$ as $\varepsilon\to 0$, with $\lambda>0$, the corresponding eigenfunctions can be approached in $L^2(-1,1)$, in the x variable, by the function:

$$\tilde u^{\varepsilon}_{i(\varepsilon)}(x)=\begin{cases}-a^{\varepsilon}\sin(k\pi x) & \text{if } x\in(-1,-\varepsilon),\\ \displaystyle\sum_{j=p(\varepsilon)}^{\infty}\alpha^{\varepsilon}_j\cos\left(\frac{j\pi x}{\varepsilon}\right) & \text{if } x\in(-\varepsilon,\varepsilon),\\ a^{\varepsilon}\sin(k\pi x) & \text{if } x\in(\varepsilon,1),\end{cases}\tag{5.8}$$

if the eigenfunction is even, or by the function

$$\tilde u^{\varepsilon}_{i(\varepsilon)}(x)=\begin{cases}a^{\varepsilon}\sin(k\pi x) & \text{if } x\in(-1,-\varepsilon)\cup(\varepsilon,1),\\ \displaystyle\sum_{j=p(\varepsilon)}^{\infty}\beta^{\varepsilon}_j\sin\left(\frac{(2j+1)\pi x}{2\varepsilon}\right) & \text{if } x\in(-\varepsilon,\varepsilon),\end{cases}\tag{5.9}$$

if the eigenfunction is odd. In formulas (5.8) and (5.9), $a^{\varepsilon}=\mathrm{O}_s(1)$ if $\lambda=(k\pi)^2$ for some k, and $a^{\varepsilon}=0$ otherwise, α^{ε}_j and β^{ε}_j are the coefficients of the expansion of $u^{\varepsilon}_{i(\varepsilon)}|_{H^1(-\varepsilon,\varepsilon)}$ in Fourier series of the eigenfunctions of the local problem (5.6):

$$\alpha^{\varepsilon}_j=2A^{\varepsilon}\frac{\sqrt{\lambda^{\varepsilon}}\,\varepsilon^{m/2-1}\sin(\varepsilon^{1-m/2}\sqrt{\lambda^{\varepsilon}})\cos(j\pi)}{\lambda^{\varepsilon}-\pi^2j^2\varepsilon^{m-2}},$$

$$\beta^{\varepsilon}_j=8B^{\varepsilon}\frac{\sqrt{\lambda^{\varepsilon}}\,\varepsilon^{m/2-1}\cos(\varepsilon^{1-m/2}\sqrt{\lambda^{\varepsilon}})\cos(j\pi)}{-4\lambda^{\varepsilon}+\pi^2(2j+1)^2\varepsilon^{m-2}},$$

respectively and $p(\varepsilon)$ is a function converging to ∞ when $\varepsilon\to 0$. In fact, the approaches (5.8) and (5.9) hold in the topology of H^1 in the macroscopic (microscopic, respectively) variable in $(-1,-\varepsilon)\cup(\varepsilon,1)$ (in $(-\varepsilon,\varepsilon)$, respectively).

Remark 5.1. – We observe that (5.8) and (5.9) confirm the strongly oscillatory character of all the eigenfunctions in $(-\varepsilon,\varepsilon)$ when $\varepsilon\to 0$, already noted in (5.4) and (5.5). In addition, the approach using the Fourier expansions is better than this by 0. Indeed, when $\lambda^{\varepsilon}_{i(\varepsilon)}$ converge towards $(k\pi)^2$ (an eigenvalue of (5.7)), then

$$\|u^{\varepsilon}_{i(\varepsilon)}\|^2_{L^2(-\varepsilon,\varepsilon)}=\mathrm{O}(\varepsilon^m).$$

For ε ranging in certain sequences ε_n (for example, ε_n such that $\sin(\sqrt{\lambda^{\varepsilon}_{i(\varepsilon)}}\,\varepsilon^{1-m/2})=0$) we can prove:

$$\|u^{\varepsilon}_{i(\varepsilon)}\|^2_{L^2(-\varepsilon,\varepsilon)}=\mathrm{O}(\varepsilon^{m+1}),\quad \text{and}\quad \|u^{\varepsilon}_{i(\varepsilon)}-\tilde{u}^{\varepsilon}_{i(\varepsilon)}\|^2_{L^2(-\varepsilon,\varepsilon)}=\mathrm{O}(\varepsilon^{\frac{5m+2}{4}}).$$

Compare with Remark 4.2 and Theorem 6.1.

Remark 5.2. – We observe that the possible $\lambda=\lambda^{\varepsilon}$ such that $\cos(\sqrt{\lambda^{\varepsilon}}\varepsilon^{1-m/2})=0$ and $\sin(\sqrt{\lambda^{\varepsilon}}(1-\varepsilon))=0$ are excluded in the equation (5.2). Each one of these values $\lambda^{\varepsilon}=(k\pi/(1-\varepsilon))^2$ with $k\in\mathbb{Z}$ is an eigenvalue of (5.1) only for certain values of ε: those satisfying the equation $2k/(2n+1)=\varepsilon_n^{m/2-1}(1-\varepsilon_n)$. In this case, the corresponding eigenfunctions are the functions (5.4) where $\lambda^{\varepsilon_n}=(k\pi/(1-\varepsilon_n))^2$, $A^{\varepsilon_n}=\mathrm{O}(\varepsilon_n^{m/2})$, and $\alpha^{\varepsilon_n}=\pm\varepsilon_n^{-m/2}$ (depending on the values of k and n). This result reaffirms approximation (5.8). Similars results are obtained when $\cos(\sqrt{\lambda^{\varepsilon}}\varepsilon^{1-m/2})=0$ and $\cos(\sqrt{\lambda^{\varepsilon}}(1-\varepsilon))=0$, excluded from (5.3). In this case, ε_n satisfy $(2k+1)/(2n+1)=\varepsilon_n^{m/2-1}(1-\varepsilon_n)$ and the corresponding eigenfunctions are the functions (5.5) with

$$\lambda^{\varepsilon_n}=\left(\frac{(2k+1)\pi}{2(1-\varepsilon_n)}\right)^2,\quad B^{\varepsilon_n}=\mathrm{O}(\varepsilon_n^{\frac{m-1}{2}}),\quad \text{and}\quad \beta^{\varepsilon_n}=\pm 1$$

(depending on the values of k and n).

6. Correcting terms for the eigenfunctions

In this section we obtain correcting terms for certain of the eigenfunctions u^{ε} of (2.2) associated with eigenvalues $\lambda^{\varepsilon}=\mathrm{O}(1)$. For the sake of simplicity we assume that the space dimension is $N=2$ and B is a circle, but calculations may be performed in other cases. We use the method of matched asymptotic expansions to compute the correcting term and the Lemma 2.3 to obtain estimates.

We consider asymptotic expansions for the eigenfunctions u^{ε} that take into account the wavelength of the vibration in εB, different from that in [8] (see Remark 4.2). Here, we outline the technique.

We postulate an asymptotic expansion of the eigenvalues λ^{ε}, $\lambda^{\varepsilon}=\lambda^*+\mathrm{o}(1)$; and of the corresponding eigenfunctions u^{ε}, an outer expansion for $x\in\Omega-\{0\}$:

$$u^{\varepsilon}(x)=u^*(x)+\mathrm{o}(1),$$

and a local expansion in a neighbourhood of $x=0$ in the local variable $y=x/\varepsilon$:

$$u^{\varepsilon}(y)=\alpha V(y)+\mathrm{o}(1),$$

for α a certain constant.

Usual techniques of matched asymptotic expansions lead us to consider the composite expansion of the eigenfunction u^{ε} in Ω:

$$u^{\varepsilon}\approx u^*(x)+\left(V^{\varepsilon}\left(\frac{x}{\varepsilon}\right)-1\right)u^*(x)\quad \text{when } u^*(0)\neq 0, \tag{6.1}$$

$$u^{\varepsilon}\approx u^*(x)+V^{\varepsilon}\left(\frac{x}{\varepsilon}\right)\quad \text{when } u^*(0)=0 \text{ and } u^*\not\equiv 0, \tag{6.2}$$

and

$$u^\varepsilon \approx V^\varepsilon\left(\frac{x}{\varepsilon}\right) \quad \text{when } u^* \equiv 0, \tag{6.3}$$

where u^* is an eigenfunction associated with the eigenvalue λ^* of the Dirichlet problem (2.8) in (6.1) and (6.2), and $V^\varepsilon(y)$ satisfies the equations:

$$\begin{cases} -\Delta_y V = \dfrac{\lambda^*}{\varepsilon^{m-2}} V & \text{in } B, \\ -\Delta_y V = 0 & \text{in } \mathbb{R}^2 - \overline{B}, \\ [V] = \left[\dfrac{\partial V}{\partial n_y}\right] = 0 & \text{on } \Gamma, \\ V(y) \to K, & \text{as } |y| \to \infty, \end{cases} \tag{6.4}$$

with $K = 0$ for (6.2) and (6.3) and with $K = 1$ for (6.1).

Regarding the existence of a nontrivial solution of (6.4), we consider $\mu = \lambda^*/\varepsilon^{m-2}$ in the first equation; then, problem (6.4) coincides with the eigenvalue problem (2.4) for a certain normalization of the eigenfunctions U and it has nontrivial solution only for certain values of ε, those values such that μ is an eigenvalue μ_k of (2.4). Therefore, formulas (6.1), (6.2) and (6.3) hold for ε such that $\lambda^*/\varepsilon^{m-2}$ is an eigenvalue of (2.4). We calculate explicitly these values in the case when B is a circle in Section 6.1 and justify formulas (6.1)–(6.3) in Section 6.2.

6.1. Eigenelements of the local problem

Let us consider problem (2.4) when $N = 2$ and B the unit circle. In order to obtain the formulas for the eigenvalues and eigenfunctions we consider polar coordinates (r, θ): $y_1 = r\cos\theta$, $y_2 = r\sin\theta$. We write indifferently $U(r,\theta)$ or $U(y)$. (2.4) becomes:

$$\begin{cases} \dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 U}{\partial \theta^2} + \mu U = 0 & \text{for } 0 < r < 1,\ 0 \leqslant \theta < 2\pi, \\ \dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 U}{\partial \theta^2} = 0 & \text{for } r > 1,\ 0 \leqslant \theta < 2\pi, \\ U|_{r=1^-} = U|_{r=1^+} \quad \text{and} \quad \dfrac{\partial U}{\partial r}|_{r=1^-} = \dfrac{\partial U}{\partial r}|_{r=1^+}, \\ U(r,\theta) \to K, \quad \text{as } r \to \infty. \end{cases} \tag{6.5}$$

We use separation of variables in (6.5) and take into account the condition of boundedness for the eigenfunctions at the point $r = 0$ and at infinity. Then, we obtain the eigenvalues and the eigenfunctions in terms of the Bessel functions of the first kind, J_n for $n = 0, 1, 2, \ldots$, and the trigonometric functions. See Section IX in [1] for an extensive exposition of properties for the Bessel functions.

Formulas (6.5) lead to the conclusion that the quantities

$$\mu_{k,n} = \nu_{k,n}^2, \quad n = 0, 1, 2, \ldots,\ k = 0, 1, 2, \ldots, \tag{6.6}$$

are the eigenvalues of (2.4), where $\nu_{k,n}$, for each fixed n, are the roots of the equation:

$$J_0'(\nu) = 0 \quad \text{when } n = 0, \quad \text{and} \quad \nu J_n'(\nu) + n J_n(\nu) = 0 \quad \text{when } n > 0. \tag{6.7}$$

To each eigenvalue $\mu_{k,0}$ there corresponds one eigenfunction (up to a constant):

$$U_{k,0}(r,\theta)=\begin{cases} J_0(\nu_{k,0}r) & \text{if } r\leqslant 1,\\ J_0(\nu_{k,0}) & \text{if } r\geqslant 1\end{cases} \tag{6.8}$$

and to each eigenvalue $\mu_{k,n},\ n>0$, there correspond two eigenfunctions (up to a constant):

$$U_{k,n}(r,\theta)=\begin{cases} J_n(\nu_{k,n}r)\cos(n\theta) & \text{if } r\leqslant 1,\\ J_n(\nu_{k,n})r^{-n}\cos(n\theta) & \text{if } r\geqslant 1\end{cases} \tag{6.9}$$

and

$$\widetilde{U}_{k,n}(r,\theta)=\begin{cases} J_n(\nu_{k,n}r)\sin(n\theta) & \text{if } r\leqslant 1,\\ J_n(\nu_{k,n})r^{-n}\sin(n\theta) & \text{if } r\geqslant 1.\end{cases} \tag{6.10}$$

As the system of the eigenfunctions $\{U_{k,0},U_{k,n},\widetilde{U}_{k,n},\ k,n=1,2,\ldots\}$ is orthogonal in $L^2(B)$, we prove that it is complete in $L^2(B)$ by using the results of completeness of the system of products (see Section VII.2 in [13]).

Indeed, taking into account the change to polar coordinates, we observe that $\{1,\cos(n\theta),\sin(n\theta)\}_{n=1}^{\infty}$ form a basis in the set of functions of $L^2(0,2\pi)$ with period 2π, and, for each fixed n, the system $\{J_n(\nu_{k,n}r)\}_{k=1}^{\infty}$ is complete in the space $L_\rho=\{f\in L^2(0,1)/\int_0^1 f(\rho)^2\rho\,\mathrm{d}\rho<\infty\}$. This fact, is a consequence of the completeness in L_ρ of the eigenfunctions of the singular Sturm–Liouville problem (for each fixed n):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}r}\left(r\dfrac{\mathrm{d}u}{\mathrm{d}r}\right)-\dfrac{n^2}{r}u+\mu r u=0 & \text{for } 0<r<1,\\ u,u' \text{ bounded when } r<1, & \\ \dfrac{\mathrm{d}u}{\mathrm{d}r}(1)+nu(1)=0. & \end{cases} \tag{6.11}$$

Simple computations allow us to prove that the whole eigenfunctions of (6.11) are $\{J_n(\nu_{k,n}r)\}_{k=1}^{\infty}$.

Therefore, the formulas (6.6) and (6.8)–(6.10) exhaust the totality of the eigenvalues and eigenfunctions of the local problem (2.4). We observe that the only eigenfunctions converging towards some constant different from zero, when $r\to\infty$, are those associated with the Bessel functions of order 0: $U_{k,0}$, $\forall k$.

6.2. Estimates for the eigenfunctions

Let us consider $\lambda^\varepsilon=\lambda^*+\mathrm{o}(1)$ for any $\lambda^*>0$. Let ε be ranging in the sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ defined by:

$$\varepsilon_k=\left(\frac{\lambda^*}{\mu_{k,0}}\right)^{\frac{1}{m-2}} \quad \text{when } \lambda^*\in\sigma_{g,0} \tag{6.12}$$

and

$$\varepsilon_k=\left(\frac{\lambda^*}{\mu_{k,n}}\right)^{\frac{1}{m-2}} \quad \text{when } \lambda^*\notin\sigma_{g,0}, \tag{6.13}$$

where $\sigma_{g,0}$ is the subset of the spectrum of the Dirichlet problem (2.8), σ_g, such that the corresponding eigenfunctions do not vanish at 0; $\sigma_{g,0}$ is not empty. Obviously, $\varepsilon_k \to 0$ as $k \to \infty$. In (6.13) n is fixed but it can take any value $n = 1, 2, \ldots$.

For fixed k, when $\lambda^* \in \sigma_{g,0}$, let ε_k be given by (6.12); we consider $V_k(y) = V^{\varepsilon_k}(y)$ the solution of (6.4) that converges towards 1 when $r \to \infty$: $V_k(y) = U_{k,0}(y)/J_0(\mu_{k,0})$ (in fact $V_k \equiv 1$ outside B). We denote by $\mathcal{T}_x$ the change of the variable from y to x, $x = y\varepsilon_k$: $\mathcal{T}_x V_k(x) = V_k(x/\varepsilon_k)$.

When $\lambda^* \notin \sigma_{g,0}$, let ε_k be given by (6.13); we consider $W_k(y) = V^{\varepsilon_k}(y)$ the solution of (6.4) that converges towards 0 when $r \to \infty$: $W_k(y) = U_{k,n}(y)$ or $W_k(y) = \widetilde{U}_{k,n}(y)$. $\mathcal{T}_x W_k$ is the function $\mathcal{T}_x W_k(x) = W_k(x/\varepsilon_k)$. We denote by ψ any smooth function taking the value 1 for $|x| < R_1$ and 0 for $|x| > R_2$, R_1 and R_2 are two fixed constants such that $R_1 < R_2$ and $B(0, R_2) \subset \Omega$.

The properties of the Bessel functions (see Sections IX and XI in [1]), the fact that, for fixed n, $\nu_{k,n} \to \infty$ when $k \to \infty$, and cumbersome calculations in polar coordinates lead us to the following estimates that will prove useful in the proofs of Theorems 6.1 and 6.2 respectively:

$$\|\nabla_y V_k\|_{L^2(B)} = \sqrt{\pi}\,\nu_{k,0} \xrightarrow{k\to\infty} \infty, \tag{6.14}$$

$$\|\nabla_y W_k\|_{L^2(B(0,R_2/\varepsilon_k)-\overline{B(0,R_1/\varepsilon_k)})} \leqslant C_1(n)\varepsilon_k^n J_n(\nu_{k,n}), \tag{6.15}$$

$$\|W_k\|_{L^2(B(0,R_2/\varepsilon_k)-\overline{B})} \leqslant C_2(n)\frac{J_n(\nu_{k,n})}{\sqrt{\varepsilon_k}}, \tag{6.16}$$

and, for $\nu_{k,n}$ large enough,

$$\|\nabla_y W_k\|_{L^2(B)} \geqslant C_3(n), \tag{6.17}$$

where $C_i(n)$, $i = 1, 2, 3$, are constants independent of k.

We prove formula (6.1) ((6.3) and (6.2) respectively) for $\varepsilon = \varepsilon_k$ as stated in Theorem 6.1 (Theorem 6.2 and Remark 6.2 respectively).

THEOREM 6.1. – *Let $\lambda^* \in \sigma_g$ and u^* an eigenfunction of* (2.8) *associated with λ^*, u^* with norm* 1 *in $H_0^1(\Omega)$ and $u^*(0) \neq 0$. Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be defined by* (6.12). *Then, there is a sequence δ^{ε_k}, $\delta^{\varepsilon_k} \to 0$ as $k \to \infty$, such that the interval $[\lambda^* - \delta^{\varepsilon_k}, \lambda^* + \delta^{\varepsilon_k}]$ contains eigenvalues of* (2.2). *Moreover, there is u^{ε_k}, u^{ε_k} with norm* 1 *in $H_0^1(\Omega)$, u^{ε_k} belonging to the eigenspace associated with the eigenvalues λ^{ε_k} in $[\lambda^* - \delta^{\varepsilon_k}, \lambda^* + \delta^{\varepsilon_k}]$ such that*

$$\left\|u^{\varepsilon_k} - \alpha^{\varepsilon_k}\big(u^* + (\mathcal{T}_x V_k - 1)u^*\big)\right\|_{H_0^1(\Omega)} \leqslant C\big(\varepsilon_k\sqrt{|\ln \varepsilon_k|}\big)^{\beta}, \tag{6.18}$$

where C and β are constants independent of ε_k, $0 < \beta < 1$, and $\alpha^{\varepsilon_k} = 1/\|(\mathcal{T}_x V_k)u^*\|_{H_0^1(\Omega)}$.

Proof. – The technique is the same as in Proposition 4.1. Here, we only outline the main steps.

Let us consider A^{ε_k} to be the operator associated with (2.8) defined by (4.1). In order to apply Lemma 2.3, it suffices to prove estimates (4.5) for $\rho^{\varepsilon_k} = \varepsilon_k\sqrt{|\ln \varepsilon_k|}$ and $\tilde{v}^{\varepsilon_k} = v^{\varepsilon_k}/\|v^{\varepsilon_k}\|_{H_0^1(\Omega)}$, being $v^{\varepsilon_k}(x) = u^*(x) + (V_k(x/\varepsilon) - 1)u^*(x) = V_k(x/\varepsilon)u^*(x)$. Then, the result in the Theorem will be satisfied for $\delta^{\varepsilon_k} = \mathrm{O}((\varepsilon_k\sqrt{|\ln \varepsilon_k|})^{1-\beta})$ and $\alpha^{\varepsilon_k} = 1/\|v^{\varepsilon_k}\|_{H_0^1(\Omega)}$.

We observe that (4.5) holds, provided that there are constants C_1 and C_2 such that for sufficiently small ε_k:

$$\left|\left\langle A^{\varepsilon_k}v^{\varepsilon_k} - \frac{1}{\lambda^*}v^{\varepsilon_k}, v\right\rangle_{H_0^1(\Omega)}\right| \leqslant C_1 R^{\varepsilon_k}\|v\|_{H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega), \tag{6.19}$$

where R^{ε_k} satisfies:

$$\frac{R^{\varepsilon_k}}{\|v^{\varepsilon_k}\|_{H_0^1(\Omega)}} \leqslant C_2 \rho^{\varepsilon_k}. \tag{6.20}$$

Subsequently, we prove estimates (6.19) and (6.20).

The definitions of A^{ε_k} and v^{ε_k} allow us to write:

$$\left\langle A^{\varepsilon_k} v^{\varepsilon_k} - \frac{1}{\lambda^*} v^{\varepsilon_k}, v \right\rangle_{H_0^1(\Omega)} = \int\limits_{\Omega-\varepsilon_k\overline{B}} u^* v\,\mathrm{d}x - \frac{1}{\lambda^*} \int\limits_{\Omega-\varepsilon_k\overline{B}} \nabla u^* . \nabla v\,\mathrm{d}x$$

$$+ \frac{1}{\varepsilon^m} \int\limits_{\varepsilon_k B} u^* (\mathcal{T}_x V_k)\, v\,\mathrm{d}x - \frac{1}{\lambda^*} \int\limits_{\varepsilon_k B} \nabla\big(u^* \mathcal{T}_x V_k\big) . \nabla v\,\mathrm{d}x, \quad \forall v \in H_0^1(\Omega). \tag{6.21}$$

When dealing with the estimates for the integrals on $\varepsilon_k B$ arising in (6.21), on account of $V_k(y)$ satisfies (6.4) in B and a Neumann condition on ∂B (see (6.8)), we apply the Green formula; we also use the inequality (see Lemma 3.2 in [15]):

$$\|v\|^2_{L^2(\varepsilon_k B)} \leqslant C_3 \varepsilon_k^2 |\ln \varepsilon_k|\, \|v\|^2_{H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega). \tag{6.22}$$

Hence, from (6.21), (6.22), the fact that (λ^*, u^*) is an eigenelement (2.8) and the Poincaré and Schwarz inequalities, we deduce (6.19) for

$$R^{\varepsilon_k} = \varepsilon_k + \Big(\varepsilon_k^{\frac{m}{2}} + \varepsilon_k \sqrt{|\ln \varepsilon_k|}\Big) \|\nabla \mathcal{T}_x V_k\|_{L^2(\varepsilon_k B)}.$$

In order to prove (6.20), we take into account that $\|u^*\|_{H_0^1(\Omega)} = 1$ and that $v^{\varepsilon_k}(x) = u^*(x)$ when $x \in \Omega - \varepsilon_k B$ and $v^{\varepsilon_k}(x) = u^*(x) V_k(x/\varepsilon_k)$ when $x \in \varepsilon_k B$; we have:

$$\big\|v^{\varepsilon_k}\big\|^2_{H_0^1(\Omega)} = 1 - \big\|\nabla u^*\big\|^2_{L^2(\varepsilon_k B)} + \big\|\nabla\big(u^* \mathcal{T}_x V_k\big)\big\|^2_{L^2(\varepsilon_k B)}. \tag{6.23}$$

Considering $R^{\varepsilon_k}/\|v^{\varepsilon_k}\|^2_{H_0^1(\Omega)}$, on account of (6.23), (6.14) and $u^*(0) \neq 0$ we obtain (6.20) and the Theorem is proved. □

THEOREM 6.2. – *Let λ^* be any positive number, $\lambda \notin \sigma_g$. Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be defined by* (6.13) *for fixed n. Then, there is a sequence δ^{ε_k}, $\delta^{\varepsilon_k} \to 0$ as $k \to \infty$, such that the interval $[\lambda^* - \delta^{\varepsilon_k}, \lambda^* + \delta^{\varepsilon_k}]$ contains eigenvalues of* (2.2). *Moreover, there is u^{ε_k}, u^{ε_k} with norm 1 in $H_0^1(\Omega)$, u^{ε_k} belonging to the eigenspace associated with all the eigenvalues λ^{ε_k} in $[\lambda^* - \delta^{\varepsilon_k}, \lambda^* + \delta^{\varepsilon_k}]$ such that*:

$$\big\|u^{\varepsilon_k} - \alpha^{\varepsilon_k}(\mathcal{T}_x W_k \psi)\big\|_{H_0^1(\Omega)} \leqslant C(n)\sqrt{\varepsilon_k}, \tag{6.24}$$

where $C(n)$ is a constant independent of ε_k and $\alpha^{\varepsilon_k} = 1/\|(\mathcal{T}_x W_k \psi)\|_{H_0^1(\Omega)}$.

Proof. – The proof is analogous to that of Theorem 6.1. On account of $\|\nabla_x u\|_{L^2(\Omega)} = \|\nabla_y u\|_{L^2(\varepsilon_k^{-1}\Omega)}$, we perform the calculations in the local variable $y = x/\varepsilon_k$.

Let $\mathbf{A}^\varepsilon$ be the positive, compact and symmetric operator on $H_0^1(\varepsilon^{-1}\Omega)$ defined by:

$$\big\langle \mathbf{A}^\varepsilon U, V\big\rangle_{H_0^1(\varepsilon^{-1}\Omega)} = \frac{1}{\varepsilon^{m-2}} \int\limits_B U V\,\mathrm{d}y + \varepsilon^2 \int\limits_{\varepsilon^{-1}\Omega - \overline{B}} U V\,\mathrm{d}y, \quad \forall U, V \in H_0^1\big(\varepsilon^{-1}\Omega\big).$$

Making the change of variable from x to y in (2.2), we obtain that the eigenvalues of $\mathbf{A}^\varepsilon$ are $1/\lambda^\varepsilon$.

Let us consider $W^{\varepsilon_k}(y) = W_k(y)\psi^{\varepsilon_k}(y)$, with $W_k = U_{k,n}(y)$ defined by (6.9) and $\psi^{\varepsilon_k}(y) = \psi(\varepsilon_k x)$ (nothing changes if W_k is defined by (6.10)).

Let us define $\widetilde{W}^{\varepsilon_k} = W^{\varepsilon_k}/\|W^{\varepsilon_k}\|_{H_0^1(\varepsilon_k^{-1}\Omega)}$. Let us admit for the time being that

$$\left\|\mathbf{A}^{\varepsilon_k}\widetilde{W}^{\varepsilon_k} - \frac{1}{\lambda^*}\widetilde{W}^{\varepsilon_k}\right\|_{H_0^1(\varepsilon_k^{-1}\Omega)} \leqslant C(n)\rho^{\varepsilon_k}, \tag{6.25}$$

where $\rho^{\varepsilon_k} = \sqrt{\varepsilon_k}J_n(\nu_{k,n})$. Then, as in Proposition 4.1, we apply Lemma 2.3 with $A = \mathbf{A}^{\varepsilon_k}$ and $H = H_0^1(\varepsilon_k^{-1}\Omega)$ and, with minor modifications, we obtain the result in the statement of the Theorem.

In order to obtain (6.25), we prove:

$$\|W^{\varepsilon_k}\|_{H_0^1(\varepsilon_k^{-1}\Omega)} \geqslant C_4(n) \tag{6.26}$$

and

$$\left|\left\langle \mathbf{A}^{\varepsilon_k}W^{\varepsilon_k} - \frac{1}{\lambda^*}W^{\varepsilon_k}, V\right\rangle_{H_0^1(\varepsilon_k^{-1}\Omega)}\right| \leqslant C_5(n)\rho^{\varepsilon_k}\|V\|_{H_0^1(\varepsilon_k^{-1}\Omega)}, \quad \forall V \in H_0^1\big(\varepsilon_k^{-1}\Omega\big), \tag{6.27}$$

for some constants $C_4(n), C_5(n)$ independent of ε_k.

Formula (6.26) is a consequence of the definition of W^{ε_k}, which takes the value (6.9) in B, and of (6.17).

In relation to (6.27), the definitions of $\mathbf{A}^{\varepsilon_k}$ and W^{ε_k} allow us to write:

$$\begin{aligned}\left\langle \mathbf{A}^{\varepsilon_k}W^{\varepsilon_k} - \frac{1}{\lambda^*}W^{\varepsilon_k}, V\right\rangle_{H_0^1(\varepsilon_k^{-1}\Omega)} &= \frac{1}{\varepsilon_k^{m-2}}\int_B W_k V\,\mathrm{d}y - \frac{1}{\lambda^*}\int_{\mathbb{R}^2}\nabla_y W_k.\nabla_y V\,\mathrm{d}y \\ &+ \frac{1}{\lambda^*}\int_{\mathbb{R}^2-\overline{B(0,R_1/\varepsilon_k)}}\nabla_y W_k.\nabla_y V\,\mathrm{d}y - \frac{1}{\lambda^*}\int_{B(0,R_2/\varepsilon_k)-\overline{B(0,R_1/\varepsilon_k)}}\nabla_y\big(W_k\psi^{\varepsilon_k}\big).\nabla_y V\,\mathrm{d}y \\ &+ \varepsilon_k^2\int_{B(0,R_1/\varepsilon_k)-\overline{B}} W_k V\,\mathrm{d}y + \varepsilon_k^2\int_{B(0,R_2/\varepsilon_k)-\overline{B(0,R_1/\varepsilon_k)}} W_k\psi^{\varepsilon_k}V\,\mathrm{d}y,\end{aligned}$$

for any $V \in H_0^1(\varepsilon_k^{-1}\Omega)$. We take into account that W_k satisfies (6.4) with $K = 0$; so that formula (2.5) for $\mu = \lambda^*/\varepsilon_k^{m-2}$ leads us to cancel the first two integrals. For the other integrals, we apply the Schwarz and Poincaré inequalities, we take into account the boundedness of ψ^{ε_k} and its derivatives, and relations (6.15) and (6.16), and then we obtain (6.27). Therefore, the Theorem is proved. □

Remark 6.1. – The result in Theorem 6.1 (6.2, respectively) allows us to assert that $(V^{\varepsilon_k}(x/\varepsilon_k) - 1)u^*(x)$, ε_k defined by (6.12) (V^{ε_k}, ε_k defined by (6.13), respectively) provides a correcting term for certain eigenfunctions u^{ε_k} of (2.2), which are approached by the eigenfunctions of the Dirichlet problem (by 0, respectively) when $\lambda^* \in \sigma_{g,0}$, i.e., when λ^* is an eigenvalue of (2.8) and the corresponding eigenfunction satisfies $u^*(0) \neq 0$ ($\lambda^* \notin \sigma_g$, respectively). In particular, estimates (6.18) and (6.24) improve the results in Theorems 4.1 and 4.2 (see also Proposition 4.1 and Remark 4.2).

Remark 6.2. – In the case when $\lambda^* \in \sigma_g$ and the corresponding eigenfunction satisfies $u^*(0) = 0$, we can prove (6.2) in a similar way to that in Theorems 6.1 and 6.2 with a suitable modification of V^ε. For example, it can be easily proved:

$$\left\| u^{\varepsilon_k} - \alpha^{\varepsilon_k}\left(v^{\varepsilon_k} + \mathcal{T}_x W_k \psi\right) \right\|_{H_0^1(\Omega)} \leqslant C\left(\sqrt{|\ln \varepsilon_k|}\right)^{\beta}, \tag{6.28}$$

where C and β are constants independent of ε_k, $0 < \beta < 1$, $\alpha^{\varepsilon_k} = 1/\|v^{\varepsilon_k} + \mathcal{T}_x W_k \psi\|_{H_0^1(\Omega)}$, v^{ε_k} is the function defined in (4.4) and $W_k\psi$ the same as in Theorem 6.2. Nevertheless, we also observe that the bound (6.28) does not improve that in Proposition 4.1 (see also Remarks 4.1 and 4.2).

Remark 6.3. – It should be noted that the results in Theorems 6.1 and 6.2 prove that certain of the eigenfunctions of (2.2) are strongly oscillating in $\varepsilon_k B$, as is the case for $J_0(\nu_{k,0} r)$ and $J_n(\nu_{k,n} r)$ for fixed n. It could also occur that the eigenfunctions u^ε of (2.2) only concentrate on the boundary of εB. In this case, in order to obtain a correcting term for u^ε, it will likely become essential to look for the so called *whispering gallery eigenfunctions* of (2.4) (see Section VII of [3] for this effect in bounded domains).

7. The case $m = 2$ and $N = 2$

In this section we study the asymptotic behaviour of the eigenelements of (2.2) when $m = 2$ and $N = 2$. As estimates (2.3) still hold for $m = 2$, the two sequences of eigenvalues associated with the global and local vibration are the same order of magnitude $O(1)$. Formal asymptotic expansions for the eigenelements have been considered in [8,15,18]; we justify here the results in these papers (see Section III.5 of [16] for another different technique when $N = 3$).

Let us consider the problem (2.2) when $m = N = 2$. Let the eigenvalues be $\{\lambda_i^\varepsilon\}_{i=1}^\infty$. Let the corresponding eigenfunctions be $\{u_i^\varepsilon\}_{i=1}^\infty$, satisfying:

$$\int_{\Omega - \varepsilon \overline{B}} u_i^\varepsilon u_j^\varepsilon \, \mathrm{d}x + \frac{1}{\varepsilon^2} \int_{\varepsilon B} u_i^\varepsilon u_j^\varepsilon \, \mathrm{d}x = \delta_{ij}, \tag{7.1}$$

where δ_{ij} is the Kronecker symbol. We prove here that the eigenvalues and eigenfunctions are approached, when $\varepsilon \to 0$, by those of problems (2.4) and (2.8). We gather the eigenelements of both problems as the eigenelements of problem:

$$\begin{cases} -\Delta_x u = \Lambda u, & u \in H_0^1(\Omega), \\ -\Delta_y U = \Lambda \chi_B(y) U, & U \in \mathcal{V}, \end{cases} \tag{7.2}$$

where χ_B is the characteristic function of B.

Problem (7.2) is a standard eigenvalue problem in the Hilbert space $L^2(\Omega) \times L^2(B)$ (its elements being pairs of functions $(u(x), U(y))$); it has an equivalent variational formulation:

Find Λ, $(u, U) \in H_0^1(\Omega) \times \mathcal{V}$, $(u, U) \neq 0$, satisfying

$$\int_\Omega \nabla_x u . \nabla_x v \, \mathrm{d}x + \int_{\mathbb{R}^2} \nabla_y U . \nabla_y V \, \mathrm{d}y = \Lambda \left[\int_\Omega u v \, \mathrm{d}x + \int_B U V \, \mathrm{d}y \right], \quad \forall (v, V) \in H_0^1(\Omega) \times \mathcal{V},$$

where $\mathcal{V}$ is the space defined in (2.6).

Let us consider

$$0 = \Lambda_1 \leqslant \Lambda_2 \leqslant \cdots \leqslant \Lambda_n \leqslant \cdots \xrightarrow{n\to\infty} \infty,$$

the sequence of eigenvalues, with the classical convention of repeated eigenvalues and let $\{(u_i, U_i)\}_{i=1}^{\infty}$ be the corresponding sequence of eigenfunctions, forming an orthonormal basis of $L^2(\Omega) \times L^2(B)$.

THEOREM 7.1. – *For each i, the i-th eigenvalue of problem* (2.2), λ_i^ε, *converges when $\varepsilon \to 0$ towards the i-th eigenvalue of* (7.2), Λ_i. *Thus, the eigenvalues Λ of* (7.2) *are the only accumulation points of $\{\lambda^\varepsilon\}_\varepsilon$ and there is conservation of the total multiplicity for ε sufficiently small.*

Proof. – Taking into account (2.3), the orthonormality condition (7.1), and the fact that $(\lambda_i^\varepsilon, u_i^\varepsilon)$ is an eigenelement of (2.2), we can extract a subsequence (still denoted by ε) such that for each $i = 1, 2, \ldots, (\lambda_i^\varepsilon, u_i^\varepsilon)$ satisfies:

$$\lambda_i^\varepsilon \xrightarrow{\varepsilon\to 0} \widetilde{\Lambda}_i, \quad \text{and} \quad \bigl(u_i^\varepsilon, U_i^\varepsilon\bigr) \xrightarrow{\varepsilon\to 0} \bigl(\tilde{u}_i, \widetilde{U}_i\bigr), \quad \text{weakly in } H_0^1(\Omega) \times \mathcal{V},$$

where $U_i^\varepsilon(y) = u_i^\varepsilon(\varepsilon y)$. Moreover, (7.1) allows us to assert that $\{(\tilde{u}_i, \widetilde{U}_i)\}_{i=1}^{\infty}$ are orthonormal in $L^2(\Omega) \times L^2(B)$. We prove that $(\widetilde{\Lambda}_i, (\tilde{u}_i, \widetilde{U}_i))$ is an eigenelement of (7.2).

Indeed, for each fixed i, provided that $\tilde{u}_i \neq 0$, we take limits in (2.2), as $\varepsilon \to 0$, and we obtain that $(\widetilde{\Lambda}_i, \tilde{u}_i)$ is an eigenelement of (2.8). Moreover, if we write (2.2) in the y variable and we take limits when $\varepsilon \to 0$, we obtain that $(\widetilde{\Lambda}_i, \widetilde{U}_i)$ satisfies (2.5). Hence, as $(\tilde{u}_i, \widetilde{U}_i) \neq (0,0)$, $(\tilde{u}_i, \widetilde{U}_i)$ is an eigenfunction of (7.2) associated with the eigenvalue $\widetilde{\Lambda}_i$.

In this way, we have $\{\widetilde{\Lambda}_i\colon i \in \mathbb{N}\} \subset \{\Lambda_i\colon i \in \mathbb{N}\}$ and, since the multiplicity of each eigenvalue is finite, we deduce that $\widetilde{\Lambda}_i \to \infty$ as $i \to \infty$. In what follows, we prove by induction that for each i, $\widetilde{\Lambda}_i = \Lambda_i$, which shows the conservation of the multiplicity.

First, let us prove the result for $i = 1$. For each $\varepsilon > 0$, let us consider $\phi^\varepsilon \in H_0^1(\Omega)$ the solution of

$$\begin{aligned} &\int_\Omega \nabla\phi^\varepsilon . \nabla v \,\mathrm{d}x + \int_{\Omega - \varepsilon\overline{B}} \phi^\varepsilon v \,\mathrm{d}x + \frac{1}{\varepsilon^2}\int_{\varepsilon B} \phi^\varepsilon v \,\mathrm{d}x \\ &\quad = (\Lambda_1 + 1)\Biggl[\int_{\Omega - \varepsilon\overline{B}} u_1 v \,\mathrm{d}x + \frac{1}{\varepsilon^2}\int_{\varepsilon B} U_1\Bigl(\frac{x}{\varepsilon}\Bigr) v(x)\,\mathrm{d}x\Biggr], \quad \forall v \in H_0^1(\Omega), \end{aligned} \tag{7.3}$$

and let Φ^ε be the function $\Phi^\varepsilon(y) = \phi^\varepsilon(\varepsilon y)$. Since (u_1, U_1) has norm equal 1 in $L^2(\Omega) \times L^2(B)$, the sequence $(\phi^\varepsilon, \Phi^\varepsilon)$ is bounded in $H_0^1(\Omega) \times \mathcal{V}$ and we can extract a subsequence that converges weakly in $H_0^1(\Omega) \times \mathcal{V}$. Taking limits in (7.3), we obtain that ϕ^ε converge towards u_1 in $L^2(\Omega)$ as $\varepsilon \to 0$. On the other hand, we can write (7.3) in the y variable and pass to the limit when $\varepsilon \to 0$ to obtain that Φ^ε converge towards U_1 in $L^2(B)$ as $\varepsilon \to 0$. Hence, denoting by $\mathcal{R}^\varepsilon(v)$ the Rayleigh quotient:

$$\mathcal{R}^\varepsilon(v) = \frac{\int_\Omega |\nabla_x v|^2 \,\mathrm{d}x + \int_{\Omega - \varepsilon\overline{B}} |v|^2 \,\mathrm{d}x + \varepsilon^{-2}\int_{\varepsilon B} |v|^2 \,\mathrm{d}x}{\int_{\Omega - \varepsilon\overline{B}} |v|^2 \,\mathrm{d}x + \varepsilon^{-2}\int_{\varepsilon B} |v|^2 \,\mathrm{d}x},$$

we have $\mathcal{R}^\varepsilon(\phi^\varepsilon) \xrightarrow{\varepsilon\to 0} \Lambda_1 + 1$. The minimax principle allows us to write $\lambda_1^\varepsilon + 1 \leqslant \mathcal{R}^\varepsilon(\phi^\varepsilon)$ and, taking limits, $\widetilde{\Lambda}_1 + 1 \leqslant \Lambda_1 + 1$. Thus, the result $\widetilde{\Lambda}_i = \Lambda_i$, holds for $i = 1$.

Let us assume that $\widetilde{\Lambda}_i = \Lambda_i$, holds for $i \leqslant j$. Taking into account that the eigenfunctions $\{(\tilde{u}_i, \widetilde{U}_i)\}_{i=1}^{j+1}$, associated with $\{\widetilde{\Lambda}_i\}_{i=1}^{j+1}$, are orthonormal in $L^2(\Omega) \times L^2(B)$, we obtain $\widetilde{\Lambda}_{j+1} \geqslant \Lambda_{j+1}$. In order to prove $\widetilde{\Lambda}_{j+1} \leqslant \Lambda_{j+1}$ we use a process of orthogonalization (see Section III.9.1 of [2], for example, for the technique):

Let us consider $\phi^\varepsilon \in H_0^1(\Omega)$ the solution of (7.3) with Λ_1, u_1 and U_1 replaced by Λ_{j+1}, u_{j+1}^* and U_{j+1}^* respectively, where (u_{j+1}^*, U_{j+1}^*) is an eigenfunction of (7.2) associated with Λ_{j+1}, with norm 1 in $L^2(\Omega) \times L^2(B)$, and such that it is orthogonal to $(\tilde{u}_i, \widetilde{U}_i)$ in $L^2(\Omega) \times L^2(B)$ for $1 \leqslant i \leqslant j$. We use the same argument as in (7.3) to prove that

$$\left(\phi^\varepsilon, \Phi^\varepsilon\right) \xrightarrow{\varepsilon \to 0} \left(u_{j+1}^*, U_{j+1}^*\right) \quad \text{in } L^2(\Omega) \times L^2(B)$$

and

$$\mathcal{R}^\varepsilon\left(\phi^\varepsilon\right) \xrightarrow{\varepsilon \to 0} \Lambda_{j+1} + 1.$$

Now, let us define

$$\psi^\varepsilon = \phi^\varepsilon - \sum_{k=1}^{j} \left(\phi^\varepsilon, u_k^\varepsilon\right)_\varepsilon u_k^\varepsilon, \qquad \Psi^\varepsilon(y) = \psi^\varepsilon(\varepsilon y),$$

where $(u, v)_\varepsilon$ denotes the scalar product in $L^2(\Omega)$:

$$(u, v)_\varepsilon = \int_{\Omega - \varepsilon \overline{B}} u v \, \mathrm{d}x + \varepsilon^{-2} \int_{\varepsilon B} u v \, \mathrm{d}x.$$

Then, it can be easily proved the convergences

$$\left(\psi^\varepsilon - \phi^\varepsilon, \psi^\varepsilon - \phi^\varepsilon\right)_\varepsilon \xrightarrow{\varepsilon \to 0} 0 \quad \text{and} \quad \left\|\nabla_x\left(\psi^\varepsilon - \phi^\varepsilon\right)\right\|_{L^2(\Omega)} \xrightarrow{\varepsilon \to 0} 0.$$

Therefore,

$$\left(\psi^\varepsilon, \Psi^\varepsilon\right) \xrightarrow{\varepsilon \to 0} \left(u_{j+1}^*, U_{j+1}^*\right) \quad \text{in } L^2(\Omega) \times L^2(B)$$

and

$$\mathcal{R}^\varepsilon\left(\psi^\varepsilon\right) \xrightarrow{\varepsilon \to 0} \Lambda_{j+1} + 1.$$

Besides, as $(\psi^\varepsilon, u_k^\varepsilon)_\varepsilon = 0$, for $1 \leqslant k \leqslant j$, the minimax principle reads: $\lambda_{j+1}^\varepsilon + 1 \leqslant \mathcal{R}^\varepsilon(\psi^\varepsilon)$. Taking limits when $\varepsilon \to 0$ we obtain $\widetilde{\Lambda}_{j+1} + 1 \leqslant \Lambda_{j+1} + 1$, and the result $\widetilde{\Lambda}_i = \Lambda_i$ holds for any $i = 1, 2, \ldots$.

We have proved the result stated in the Theorem on the eigenvalues for a certain subsequence ε, but, taking into account that for any sequence it is possible to extract a subsequence satisfying the same result, the Theorem is proved. □

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