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ON THE WHISPERING GALLERY MODES ON INTERFACES OF MEMBRANES COMPOSED OF TWO MATERIALS WITH VERY DIFFERENT DENSITIES

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We consider a vibrating membrane occupying a domain Ω of $\mathbb{R}^2$, composed of two materials, with very different densities. These materials fill two domains Ω_1 and Ω_2 of $\mathbb{R}^2$, and Γ is the interface between them: $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. We look at the associated spectral problem. We prove that there are modes which concentrate in a small neighborhood of Γ, *the whispering gallery modes*. We address the cases where Ω_2, the part with negligible mass, is either a bounded or unbounded domain ($\Omega_2 = \mathbb{R}^2 - \bar{\Omega}_1$), and the case where Ω_1 is a *concentrated mass*: $\Omega_1 = \varepsilon B$, with $\varepsilon \to 0$, and the density in Ω_1 very much higher than elsewhere.

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1. Introduction and Statement of the Problems

This paper deals with the computation of the *whispering gallery modes on interfaces* for vibrating membranes having two components with very different densities; i.e. eigenfunctions which are asymptotically zero except in a thin layer surrounding the interface between the two components.

Let us recall that the existence of such a kind of eigenfunctions concentrating on the boundary of a homogeneous circular membrane was first discovered by Rayleigh[22] in order to explain the whispering gallery phenomenon in acoustics. Whispering gallery modes are of obvious interest in the study of the vibrations of coupled systems and the phenomena has been very well described in the literature: see Secs. IV–VII of Ref. 3, Sec. III.3 of Ref. 8, Ref. 16 and Sec. XIV.287 of Ref. 23 for references as well as for historical notes; see Secs. V and VII of Ref. 3, Refs. 7 and 15 for techniques based on the geometrical theory of diffraction. The problems and technique in this paper differ from those in the previous papers.

As is well known, the study of the vibrations of certain mechanical systems containing a part with negligible small mass leads us to the study of the spectral problem:

Find λ and $u \neq 0$ such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega_1 , \\ -\Delta u = 0 & \text{in } \Omega_2 , \\ \quad [u] = \left[\dfrac{\partial u}{\partial n}\right] = 0 & \text{on } \Gamma , \end{cases} \tag{1.1}$$

$$u = 0 \quad \text{on } \Sigma . \tag{1.2}$$

Here, Ω_1 and Ω_2 are two open bounded domains of $\mathbb{R}^2$ with smooth boundaries, Γ is the boundary of Ω_1, $\Gamma \cup \Sigma$ is the boundary of Ω_2, $\bar{n}$ is the unit outward normal to Γ and the brackets denote the jump across Γ of the enclosed quantities. We have assumed that the part of the system with negligible mass fills the outer domain Ω_2: the density takes the value 1 in Ω_1 and 0 in Ω_2; Γ is the *interface* between the two domains Ω_1 and Ω_2.

In the case where Ω_2 is the outer domain of Ω_1, $\Omega_2 = \mathbb{R}^2 - \bar{\Omega}_1$, the condition on Σ (1.2) becomes:

$$u(y) \to c \quad \text{as } |y| \to \infty , \tag{1.3}$$

where c is an unknown but well-determined constant.

We refer to Secs. IV.6 and IV.8 of Ref. 25 for the variational formulation of problems (1.1)–(1.2) and (1.1)–(1.3) in $H^1(\Omega_1)$. Both problems can be written as *standard vibration problems* with a discrete spectrum:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \cdots \xrightarrow{n\to\infty} \infty ,$$

where the classical convection of repeated index is adopted; the corresponding eigenfunctions form a basis of $L^2(\Omega_1)$ and $H^1(\Omega_1)$.

We observe that problem (1.1)–(1.2) appears, for example, in a natural way when studying vibrating membranes containing a very heavy inclusion. On the other hand, problem (1.1)–(1.3) appears as a *microscopic* or *local problem* when studying, for example, vibrating membranes with concentrated masses; i.e. the vibrations of membranes which have very heavy small inclusions (cf. (1.4)). More specifically, (1.1)–(1.3) appears involved with the low frequency vibrations for these systems with concentrated masses (see Refs. 9, 18 and Sec. VII.10 of Ref. 25). Nevertheless, it has also been proved that a study of the eigenfunctions associated with very large eigenvalues of (1.1)–(1.3) is essential in order to obtain information on the high frequency vibrations for systems with concentrated masses (see Ref. 5).

The eigenfunctions of (1.1)–(1.2) and (1.1)–(1.3) that we deal with, in Secs. 2–4 of this paper, are associated with very large eigenvalues: we focus on obtaining explicit asymptotic formulas for certain eigenvalues λ corresponding with vibrations concentrating near Γ; i.e. the corresponding eigenfunctions are significant in a small neighborhood of Γ and they almost vanish outside. In fact, we show that they are strongly oscillating functions in the direction s and decay exponentially at a very short distance of Γ in the direction n, where (s, n) are orthogonal curvilinear coordinates in a neighborhood of Γ and s is the arc. These eigenfunctions are referred to as the *whispering gallery eigenfunctions on the interface* Γ. In Sec. 5, we also show the connection of the whispering gallery eigenfunctions of (1.1)–(1.3) with vibrating membranes with concentrated masses (see problem (1.4)). The main results in this paper are in Secs. 3–5.

To be more precise, in Sec. 2 we compute explicitly the eigenvalues and eigenfunctions in the case where Ω_2 is either a circle of radius $R > 1$ or an unbounded domain; for simplicity, we consider the case where Γ is a circumference of radius 1 (see Remark 3.1). In Secs. 3 and 4 we compute the whispering gallery modes and the corresponding frequencies of (1.1)–(1.3) and (1.1)–(1.2), respectively. By means of asymptotic expansions we show that there are certain eigenvalues of (1.1)–(1.3) ((1.1)–(1.2), respectively) $\lambda = O(n^2)$, with $n \in \mathbb{N}$, $n \to \infty$ (cf. (3.1) and (4.1), respectively), such that, inside B, the corresponding eigenfunctions oscillate in a small neighborhood of Γ, of width $O(\frac{1}{n^{2/3}})$, and decay exponentially depending on the distance to Γ out of this *boundary layer of thickness* $O(\frac{1}{n^{2/3}})$. Outside B the exponential decay of the eigenfunctions also holds (cf. Eqs. (2.12), (3.8)–(3.11), and Fig. 1).

Symbols o and O are the classical Landau order symbols. Also, O_s, $\ll$ and $\sim$, appearing throughout Secs. 2–4, are some classical order symbols used for asymptotic expansions (cf. for example Sec. 1 of Refs. 4 and 8). In order to prove the previously mentioned results within the asymptotic expansions theory, in Secs. 3 and 4, we

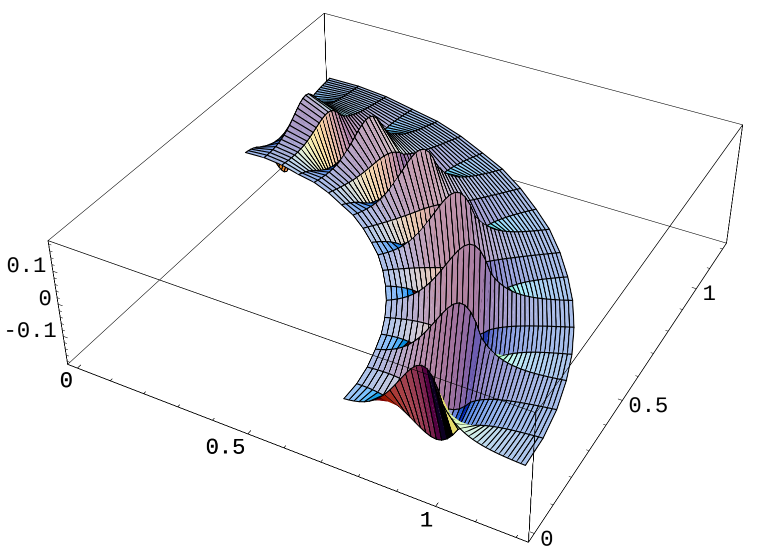

Fig. 1. Graphic of $U_{k,n}(r, \theta)$, $n = 50$, $k = 1$, $\theta \in [0, \pi/2]$, $r \in [0.75, 1.2]$.

use a general idea: to examine the modes and identify those that can support waves along the curve Γ. The computations are mainly based on the properties of the Airy and Bessel functions, and on the asymptotic behavior of the Bessel functions for large nearly equal argument and order (cf. Sec. 2.1 and Remark 3.1).

In Sec. 5 we show that there also exist eigenfunctions of the whispering gallery type, concentrating in a neighborhood of the interface, for a model of vibrating membrane with one single concentrated mass. The spectral problem being:

$$\begin{cases} -\Delta u^\varepsilon = \lambda^\varepsilon \rho^\varepsilon u^\varepsilon & \text{in } \Omega\,, \\ u^\varepsilon = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.4}$$

where Ω is a bounded domain or $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$, $\rho^\varepsilon = \rho^\varepsilon(x)$ is the density function defined by

$$\rho^\varepsilon(x) = \begin{cases} \dfrac{1}{\varepsilon^m} & \text{if } x \in B^\varepsilon \\ 1 & \text{if } x \in \Omega - \overline{B^\varepsilon}\,, \end{cases} \tag{1.5}$$

with $B^\varepsilon = \varepsilon\Omega_1$, m is a parameter fixed, $m > 2$, and ε is a small positive parameter ($\varepsilon \to 0$). For simplicity, we shall assume that Ω_1 is a circle of radius one centered at the origin that we shall denote by B.

The asymptotic behavior, as $\varepsilon \to 0$, of the eigenelements $(\lambda^\varepsilon, u^\varepsilon)$ of problem (1.4) has been extensively studied by many authors using different techniques: see, e.g. Refs. 5, 9, 18, 19 and 25 (see Refs. 10–12 and 14 dealing with homogenization problems for vibrating systems with many concentrated masses). It is well known that, for each fixed $\varepsilon > 0$, (1.4) is a standard eigenvalue problem with a discrete spectrum

$$0 < \lambda_1^\varepsilon \le \lambda_2^\varepsilon \le \cdots \le \lambda_n^\varepsilon \le \cdots \xrightarrow{n\to\infty} \infty\,,$$

where the classical convention of repeated eigenvalues has been adopted. Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the corresponding eigenfunctions, which are assumed to be an orthonormal basis in $H_0^1(\Omega)$, i.e. $\|\nabla u_i^\varepsilon\|_{L^2(\Omega)} = 1$. The minimax principle gives the estimates:

$$C\varepsilon^{m-2}|\ln\varepsilon|^{-1} < \lambda_i^\varepsilon < C_i\varepsilon^{m-2}\,, \quad \text{for each fixed } i = 1, 2, 3, \dots\,, \tag{1.6}$$

where C, C_i are constants independent of ε and $C_i \to \infty$ when $i \to \infty$.

Bounds (1.6) allow us to assert that the low frequencies λ_i^ε are of order $O(\varepsilon^{m-2})$. In addition, it has been proved (see Ref. 9), that these frequencies and the corresponding eigenfunctions, are approached through those of the local problem (1.1)–(1.3).

The eigenvalues $\lambda_{i(\varepsilon)}^\varepsilon$ of order $O(1)$ are referred to as the high frequencies of (1.4). Their asymptotic behavior and the structure of the corresponding eigenfunctions has also been addressed in Refs. 5 and 9. In this paper we complete and improve results in the previous papers.

As a matter of fact, in Ref. 5, two problems appear in a natural way associated with the high frequencies: the local problem (1.1)–(1.3) for $\Omega_1 \equiv B$, and the Dirichlet problem:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega\,, \\ \quad u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.7}$$

Roughly speaking, it has been shown that the high frequencies accumulate on the whole positive real axis and only for the eigenvalues λ^ε converging, as $\varepsilon \to 0$, towards an eigenvalue λ^0 of (1.7), the corresponding eigenfunctions u^ε do not vanish asymptotically in Ω; these eigenfunctions u^ε are approached in $H_0^1(\Omega)$ through an eigenfunction of (1.7), u^0, associated with λ^0. Besides, all the eigenfunctions u^ε associated with the high frequencies have an oscillatory character in εB: in *the local variable* $y = x/\varepsilon$, they are approached through the eigenfunctions of the local problem (1.1)–(1.3) associated with large frequencies.

In Sec. 5 we follow the idea in Ref. 5 to prove that the eigenfunctions, associated with certain high frequencies of (1.4), concentrate in a neighborhood of the interface Γ^ε of thickness $O(\varepsilon^{\frac{m+1}{3}})$ and vanish asymptotically elsewhere. In this case, ε ranges in subsequences converging towards 0 (cf. (5.1)) and the corresponding frequencies λ^ε converge towards positive values λ^*, where λ^* is not an eigenvalue of (1.7) (see Theorem 5.1 and Remark 5.1).

Finally, let us observe that whispering gallery modes, concentrating in a neighborhood of the boundary, have been explicitly computed in Sec. VII.2 of Ref. 3, for problem (1.7) when Ω is a circular domain of $\mathbb{R}^2$ (see Remark 3.1). The problems considered in this paper, (1.1)–(1.2), (1.1)–(1.3) and (1.4), as well as the technique, are different from those in Ref. 3. As regard (1.1)–(1.2) and (1.1)–(1.3) we obtain explicit formulas for the whispering gallery eigenfunctions and the corresponding eigenfrequencies. Instead, using these whispering gallery modes of the local problem (1.1)–(1.3), we obtain *quasimodes* which concentrate near the interface for systems with concentrated masses (1.4) (see Theorem 5.1 and Remarks 5.2 and 5.3).

2. The Eigenelements of (1.1)–(1.3) and (1.1)–(1.2)

We obtain explicit formulas for the eigenvalues and the eigenfunctions of (1.1)–(1.3) and (1.1)–(1.2) in terms of the Bessel functions. Besides, for the sake of completeness, in Sec. 2.1, we state certain known asymptotic formulas connected with the Bessel functions (cf. Refs. 1, 4, 20, 21 and 26).

Let B (B_R, respectively) denote the circle of radius 1 (R, respectively), centered at the origin. For simplicity, we consider problem (1.1)–(1.2) in the particular case where Ω_1 and Ω_2 are B and B_R, with $R > 1$, respectively. Similarly, we consider (1.1)–(1.3) in the case where $\Omega_1 = B$ and $\Omega_2 = \mathbb{R}^2 - \bar{B}$.

In order to obtain the formulas for the eigenvalues and eigenfunctions we consider polar coordinates (r, θ): $y_1 = r\cos\theta$, $y_2 = r\sin\theta$. We write indifferently

$U(r,\theta)$ or $U(y)$. Problem (1.1)–(1.3) reads:

$$\begin{cases} \dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 U}{\partial \theta^2} + \lambda U = 0 \quad \text{for } 0 < r < 1,\ 0 \leq \theta < 2\pi\,, \\ \qquad \dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 U}{\partial \theta^2} = 0 \quad \text{for } r > 1,\ 0 \leq \theta < 2\pi\,, \\ \qquad U|_{r=1^-} = U|_{r=1^+} \quad \text{and} \quad \left.\dfrac{\partial U}{\partial r}\right|_{r=1^-} = \left.\dfrac{\partial U}{\partial r}\right|_{r=1^+}\,, \\ \qquad\qquad U(r,\theta) \to K\,, \quad \text{as } r \to \infty\,. \end{cases} \tag{2.1}$$

Using separation of variables in (2.1), the eigenvalues and the eigenfunctions of (2.1) have been found in Ref. 5 in terms of the Bessel functions of the first kind, J_n for $n = 0, 1, 2, \ldots$, and of the trigonometric functions. We just sketch here the formulas.

The quantities

$$\mu_{k,n} = \nu_{k,n}^2\,, \quad n = 0, 1, 2, \ldots\,, \quad k = 1, 2, \ldots\,, \tag{2.2}$$

are the eigenvalues of (1.1)–(1.3), where $\nu_{k,n}$, for each fixed n, are the roots of the equation:

$$J_0'(\nu) = 0 \quad \text{when } n = 0 \quad \text{and} \quad \nu J_n'(\nu) + nJ_n(\nu) = 0 \quad \text{when } n > 0\,. \tag{2.3}$$

To each eigenvalue $\mu_{k,0}$ there corresponds one eigenfunction (up to a constant):

$$U_{k,0}(r,\theta) = \begin{cases} J_0(\nu_{k,0} r) & \text{if } r \leq 1\,, \\ J_0(\nu_{k,0}) & \text{if } r \geq 1\,, \end{cases} \tag{2.4}$$

and to each eigenvalue $\mu_{k,n}$, $n > 0$, there correspond two eigenfunctions (up to a constant):

$$U_{k,n}(r,\theta) = \begin{cases} J_n(\nu_{k,n} r)\cos(n\theta) & \text{if } r \leq 1\,, \\ J_n(\nu_{k,n}) r^{-n} \cos(n\theta) & \text{if } r \geq 1 \end{cases} \tag{2.5}$$

and

$$\tilde{U}_{k,n}(r,\theta) = \begin{cases} J_n(\nu_{k,n} r)\sin(n\theta) & \text{if } r \leq 1\,, \\ J_n(\nu_{k,n}) r^{-n} \sin(n\theta) & \text{if } r \geq 1\,. \end{cases} \tag{2.6}$$

Besides, the system of the eigenfunctions of (1.1)–(1.3), $\{U_{k,0}, U_{k,n}, \tilde{U}_{k,n}, k, n = 1, 2, \ldots\}$ is an orthogonal basis of $L^2(B)$ and $H^1(B)$.

As regard problems (1.1)–(1.2), we proceed here below as in Sec. VI of Ref. 5, with minor modifications, to obtain the orthogonal basis $\{V_{k,0}, V_{k,n}, \tilde{V}_{k,n}, k, n = 1, 2, \ldots\}$ in $L^2(B)$ and $H^1(B)$ formed by the eigenfunctions of (1.1)–(1.2).

We write problem (1.1)–(1.2) in polar coordinates, and take into account the properties of the Bessel functions and the completeness of the system of products (see Sec. IX of Ref. 1 and Sec. VII.2 of Ref. 17, respectively), to obtain the eigenvalues and eigenfunctions of (1.1)–(1.2) in terms of the Bessel functions of the first kind, J_n for $n = 0, 1, 2, \ldots$, and of the trigonometric functions.

Indeed, the quantities

$$\tilde{\mu}_{k,n} = \tilde{\nu}_{k,n}^2 \,, \quad n = 0, 1, 2, \ldots \,, \quad k = 1, 2, \ldots \,, \tag{2.7}$$

are the eigenvalues of (1.1)–(1.2), where $\tilde{\nu}_{k,n}$, for each fixed n, are the roots of the equation:

$$\begin{aligned} \nu J_0'(\nu) + \frac{1}{\ln R} J_0(\nu) = 0 \quad \text{when } n = 0 \,; \\ \nu J_n'(\nu) + \frac{R^{2n}+1}{R^{2n}-1} n J_n(\nu) = 0 \quad \text{when } n > 0 \,. \end{aligned} \tag{2.8}$$

To each eigenvalue $\tilde{\mu}_{k,0}$ there corresponds one eigenfunction (up to a constant):

$$V_{k,0}(r,\theta) = \begin{cases} J_0(\tilde{\nu}_{k,0} r) & \text{if } r \le 1 \,, \\ \dfrac{J_0(\tilde{\nu}_{k,0})}{\ln R}(\ln R - \ln r) & \text{if } r \ge 1 \,, \end{cases} \tag{2.9}$$

and, to each eigenvalue $\tilde{\mu}_{k,n}$, $n > 0$, there correspond two eigenfunctions (up to a constant):

$$V_{k,n}(r,\theta) = \begin{cases} J_n(\tilde{\nu}_{k,n} r)\cos(n\theta) & \text{if } r \le 1 \,, \\ \dfrac{J_n(\tilde{\nu}_{k,n})}{1 - R^{2n}}(r^n - R^{2n} r^{-n})\cos(n\theta) & \text{if } r \ge 1 \end{cases} \tag{2.10}$$

and

$$\tilde{V}_{k,n}(r,\theta) = \begin{cases} J_n(\tilde{\nu}_{k,n} r)\sin(n\theta) & \text{if } r \le 1 \,, \\ \dfrac{J_n(\tilde{\nu}_{k,n})}{1 - R^{2n}}(r^n - R^{2n} r^{-n})\sin(n\theta) & \text{if } r \ge 1 \,. \end{cases} \tag{2.11}$$

Therefore, we have obtained explicit formulas for the eigenfunctions of (1.1)–(1.3) and (1.1)–(1.2).

Remark 2.1. We observe that in relation to (1.1)–(1.3), formulas (2.4)–(2.6), allow us to assert that the only eigenfunctions converging towards some constant K different from zero, when $r \to \infty$, are those associated with the Bessel functions of order $0 : U_{k,0}, \forall\, k$. For $n > 0$, from (2.5) and (2.6), we also observe that, for fixed θ, the eigenfunctions decrease exponentially in the radial direction r. The same behavior holds for $n > 0$ and the eigenfunctions of problem (1.1)–(1.2) in formulas (2.10) and (2.11): it suffices to write

$$\frac{r^n - R^{2n} r^{-n}}{1 - R^{2n}} = e^{-n \ln r} \frac{R^{2n} - r^{2n}}{R^{2n} - 1} \quad \text{for } 1 < r < R \,, \tag{2.12}$$

to obtain this exponential decay.

Remark 2.2. Formulas (2.5) and (2.6) lead us to the conclusion that, outside B, the oscillations of the eigenfunctions only occur in the direction of the arc θ and not in the radial direction; an exponential decay of the eigenfunctions occurs in the radial direction (see Remark 2.1). Instead, inside B, if we consider fixed n and $\nu_{k,n}$ sufficiently large, $\nu_{k,n}$ being the zeros of Eq. (2.3), the properties of the Bessel functions $J_n(\nu)$ for fixed n, and $\nu \to \infty$ (see Sec. IX.2 of Ref. 1) allow us to assert that the eigenfunctions $U_{k,n}(x)$ and $\tilde{U}_{k,n}(x)$ are strongly oscillating functions in the radial direction. On the other hand, if n tends to ∞, one may think that the roots $\nu_{k,n}$ of (2.3) could be sufficiently small in such a way that the eigenfunctions do not exhibit any concentration in the radial direction.

Thus, in order to obtain eigenfunctions which concentrate their support in a neighborhood of Γ, we have to look for both n and $\nu_{k,n}$ converging towards ∞. In fact, as in the case of zeros of a Bessel function or its derivative (see Sec. VII of Ref. 21 and Sec. XV.3 of Ref. 26), in Sec. 3, we prove that when $n \to \infty$, for fixed k, the zeros $\nu_{k,n}$ of (2.3) are already of order $O(n)$, and they give rise to vibrations of the whispering gallery type.

The same properties hold for $V_{k,n}(x)$ and $\tilde{V}_{k,n}(x)$ in formulas (2.10) and (2.11) and the zeros $\tilde{\nu}_{k,n}$ of Eq. (2.8) respectively (see Sec. 4).

In the following Secs. 3 and 4, we are concerned with the whispering gallery modes of (1.1)–(1.3) and (1.1)–(1.2), respectively. Therefore, among all the eigenfunctions (2.4)–(2.6) ((2.9)–(2.11), respectively) we shall look for those associated with $\nu_{k,n}$ ($\tilde{\nu}_{k,n}$, respectively), roots of Eq. (2.3) ((2.8), respectively), such that the corresponding Bessel function $J_n(\nu_{k,n}r)$ exhibits an exponential decay at a small distance from the boundary Γ, i.e. an exponential decay for $r < 1$, and r near $r = 1$, since, on account of Remark 2.1, this decay already occurs for $r > 1$. According to Remark 2.2, asymptotics of the Bessel functions for large order and arguments, that we sketch in Sec. 2.1 below, are used in order to find these eigenfunctions and the corresponding $\nu_{k,n}$ ($\tilde{\nu}_{k,n}$, respectively).

2.1. *Auxiliary asymptotic formulas*

It proves to be useful for the rest of the paper to gather in this section some known properties of the Bessel functions and, consequently, of the Airy function.

In particular, in order to be self-contained, we introduce the expansions of the Bessel functions, for large orders and arguments, in terms of the Airy function Ai and its derivative Ai′ (see Sec. IX.3 of Ref. 1 and Sec. IV of Ref. 21):

$$J_n(nz) \sim \left(\frac{4\xi}{1-z^2}\right)^{1/4} \left(\frac{\mathrm{Ai}(n^{2/3}\xi)}{n^{1/3}} \sum_{s=0}^{\infty} \frac{a_s(\xi)}{n^{2s}} + \frac{\mathrm{Ai}'(n^{2/3}\xi)}{n^{5/3}} \sum_{s=0}^{\infty} \frac{b_s(\xi)}{n^{2s}}\right), \quad (2.13)$$

where $a_0(\xi) \equiv 1$, $a_s(\xi)$, $b_s(\xi)$, for $s = 0, 1, \ldots$, are regular functions of ξ (in a certain region of the complex plane), defined by certain integrals, and z and ξ are complex variables related by

$$\frac{2}{3}\xi^{3/2} = \ln\left(\frac{1+\sqrt{1-z^2}}{z}\right) - \sqrt{1-z^2}\,. \tag{2.14}$$

ξ in (2.14) is chosen to be real for positive z, and, as $n \to \infty$, the expansion (2.13) is uniform with respect to z in

$$|\arg(z)| \le \pi - \delta\,, \quad \forall\,\delta \quad \text{with } \delta > 0\,, \quad z \neq 0\,.$$

Besides, for z in a small neighborhood of $z = 1$, ξ is in a small neighborhood of $\xi = 0$ and we have the asymptotic expansions:

$$z(\xi) = 1 - \frac{\xi}{2^{1/3}} + \frac{3}{10}\left(\frac{\xi}{2^{1/3}}\right)^2 + O(\xi^3)\,, \tag{2.15}$$

that can be found in Sec. IV of Ref. 21, and

$$\xi(z) = 2^{1/3}(1-z) + \frac{3}{5}\frac{1}{2^{2/3}}(1-z)^2 + O(1-z)^3\,, \tag{2.16}$$

obtained by means of simple computations.

Similarly to (2.13), we have the asymptotic expansions for the derivatives $J'_n(nz)$:

$$J'_n(nz) \sim -\frac{2}{z}\left(\frac{4\xi}{1-z^2}\right)^{-1/4}\left(\frac{\mathrm{Ai}(n^{2/3}\xi)}{n^{4/3}}\sum_{s=0}^{\infty}\frac{c_s(\xi)}{n^{2s}} + \frac{\mathrm{Ai}'(n^{2/3}\xi)}{n^{2/3}}\sum_{s=0}^{\infty}\frac{d_s(\xi)}{n^{2s}}\right), \tag{2.17}$$

where the same notations as in (2.13) are considered, and $d_0(\xi) \equiv a_0(\xi) \equiv 1$.

The Airy function in (2.13) and (2.17) is a well-known solution of

$$\frac{d^2\mathrm{Ai}}{dt^2} = t\mathrm{Ai}\,. \tag{2.18}$$

Its zeros, denoted by a_k, $k = 1,2,3,\dots(\mathrm{Ai}(a_k) = 0)$, are strictly negative real numbers, $a_k \to -\infty$ as $k \to \infty$.

As regards to the positive zeros $\hat{\nu}_{k,n}$ of the Bessel functions $J_n(\nu)$, as $n \to \infty$, we have the asymptotics:

$$\hat{\nu}_{k,n} = n\left(1 - \frac{a_k}{2^{1/3}n^{(2/3)}} + O\left(\frac{1}{n^{4/3}}\right)\right), \quad k = 1,2,3,\dots\,, \tag{2.19}$$

where a_k is the kth zero of the Airy function. Besides, the asymptotics for J'_n in the zeros of J_n, for large n, gives:

$$J'_n(\hat{\nu}_{k,n}) = O_s(n^{-(2/3)})\,. \tag{2.20}$$

See Secs. IV and VII of Ref. 21 and Sec. IX.5 of Ref. 1 for (2.19) and (2.20).

For convenience, we introduce here the estimates for the Airy function and its derivative:

$$|\mathrm{Ai}(z)| \le C(1+|z|^{1/4})^{-1}|e^{-(2/3)z^{3/2}}| \quad \text{and} \quad |\mathrm{Ai}'(z)| \le C(1+|z|^{1/4})|e^{-(2/3)z^{3/2}}|\,, \tag{2.21}$$

where C is some positive constant and $|\arg(z)| \leq \pi$. We also introduce the asymptotic expansions of the Airy function and its derivative for large positive arguments

$$\mathrm{Ai}(t) = \frac{1}{2}\pi^{-1/2}t^{-1/4}e^{-(2/3)t^{3/2}}(1 + O(t^{-3/2}))\,, \quad \text{for } t \to +\infty\,, \tag{2.22}$$

$$\mathrm{Ai}'(t) = -\frac{1}{2}\pi^{-1/2}t^{1/4}e^{-(2/3)t^{3/2}}(1 + O(t^{-3/2}))\,, \quad \text{for } t \to +\infty\,, \tag{2.23}$$

and, for large negative arguments

$$\mathrm{Ai}(-t) = \pi^{-1/2}t^{-1/4}\left(\cos\left(\frac{2}{3}t^{3/2} - \frac{1}{4}\pi\right) + O(t^{-3/2})\right), \quad \text{for } t \to +\infty\,, \tag{2.24}$$

$$\mathrm{Ai}'(-t) = \pi^{-1/2}t^{1/4}\left(\cos\left(\frac{2}{3}t^{3/2} - \frac{3}{4}\pi\right) + O(t^{-3/2})\right), \quad \text{for } t \to +\infty\,. \tag{2.25}$$

We refer, e.g., to Sec. X.4 of Ref. 1, Sec. IV.6 of Ref. 4 and Sec. VI.4 of Ref. 26 for the definition of the Airy function, its properties and its connection with the Bessel functions. We refer to Secs. IX.3 and X.4 of Ref. 1, Sec. IV of Ref. 20 and Appendix in Ref. 21 for the more specific properties of the Airy function (2.21)–(2.25).

3. Whispering Gallery Eigenelements of (1.1)–(1.3)

In this section we provide asymptotic expansions for whispering gallery eigenfunctions on the interface Γ and for the corresponding frequencies of problem (1.1)–(1.3). First, we obtain asymptotics (3.1) for the eigenvalues $\mu_{k,n} = \nu_{k,n}^2$; then, in Sec. 3.1, we obtain asymptotics (3.9)–(3.11) for the eigenfunctions which allow us to assert that, inside B, the eigenfunctions associated with $\nu_{k,n}$ in (3.1) decay exponentially out of a boundary layer of width $O(\frac{1}{n^{2/3}})$. Finally, in Sec. 3.2, we obtain some energy estimates.

Throughout the sections, we consider that n tends to ∞. Moreover, if there is no possibility of error, we use the symbol O as follows: for f and g functions depending on $n \to \infty$ (similarly, $\xi \to 0$), $f = O(g)$ when $f = gC + o(g)$, with C some constant different from zero.

In the following lemma, we obtain the asymptotics for the eigenvalues of (1.1)–(1.3). For the sake of completeness of this section, here below, we use the asymptotics (2.19) to prove Lemma 3.1. This lemma can also be deduced from the more general result in Proposition 4.2 of Sec. 4 (see Remark 4.1).

Lemma 3.1. *Let $\nu_{k,n}$ be the zeros of* (2.3). *Then, for $k = 1, 2, 3, \ldots$, the expansion*

$$\nu_{k,n} = n\left(1 - \frac{a_k}{2^{1/3}n^{2/3}} - \frac{1}{n} + O\left(\frac{1}{n^{4/3}}\right)\right) \tag{3.1}$$

holds, as $n \to \infty$, where a_k is the kth zero of the Airy function Ai *in* (2.18).

Proof. Expansion (3.1) is obtained as a consequence of (2.3), of the recurrence relations for the Bessel functions (cf., e.g., Sec. III.13 of Ref. 26 and Sec. IX.1 of

Ref. 1)

$$nJ_n(\nu) + \nu J_n'(\nu) = \nu J_{n-1}(\nu)\,, \tag{3.2}$$

and of (2.19); we already have

$$\nu_{k,n} = \hat{\nu}_{k,n-1} = (n-1)\left(1 - \frac{a_k}{2^{1/3}(n-1)^{2/3}} + O\left(\frac{1}{(n-1)^{4/3}}\right)\right), \quad k = 1, 2, 3, \ldots,$$

and then, we have proved (3.1). □

3.1. *On the whispering gallery eigenfunctions*

We prove that inside B the eigenfunctions $U_{k,n}(r,\theta)$ and $\tilde{U}_{k,n}(r,\theta)$ associated with the eigenvalues $\mu_{k,n} = \nu_{k,n}^2$, $\nu_{k,n}$ in (3.1), and k fixed ($k = 1, 2, 3, \ldots$), oscillate for $1 - O(\frac{1}{n^{2/3}}) < r < 1$ and decrease exponentially, depending on the distance to the interface Γ, out of this layer (see (3.9)–(3.11)). As a consequence of the exponential decay of these eigenfunctions when $n \to \infty$ for $r > 1$ (see Remark 2.1), we can assert that, asymptotically, they concentrate their support in a thin layer of width $O(\frac{1}{n^{2/3}})$ near Γ (cf. Fig. 1). Also, the energy estimates in Sec. 3.2 lead us to the conclusion that most of the energy, of these whispering gallery eigenfunctions, concentrates in the neighborhood Γ of width $O(\frac{1}{n^{2/3}})$ (see Remark 3.2).

Considering (2.5) and (2.6) (see Remarks 2.1 and 2.2), we focus on obtaining the exponential decay as $r < 1$ for the Bessel functions $J_n(\nu_{k,n} r)$: for $\nu_{k,n}$ given by (3.1), we prove formulas (3.9)–(3.11) at the end of this section.

For fixed $k = 1, 2, 3, \ldots$, let us denote

$$\nu_{k,n} = n(1 + \nu^*(n))$$

where, on account of (3.1),

$$\nu^*(n) = -\frac{a_k}{2^{1/3} n^{2/3}} - \frac{1}{n} + O\left(\frac{1}{n^{4/3}}\right), \tag{3.3}$$

is a positive number for n sufficiently large.

Considering

$$z = \frac{\nu_{k,n}}{n} r\,, \tag{3.4}$$

(3.1) and (3.3), we have

$$z = (1 + \nu^*(n))r\,, \quad \text{with } \nu^*(n) = O_s(n^{-2/3}) \quad \text{and} \quad \nu^*(n) > 0\,. \tag{3.5}$$

Thus, when r is in a small neighborhood of 1, z and ξ are in a small neighborhood of 1 and 0 respectively, and (2.16) reads:

$$\xi(z) = 2^{1/3}(1 - (1 + \nu^*(n))r) + O((1 - (1 + \nu^*(n))r)^2)\,. \tag{3.6}$$

Besides, for $0 < r < 1$, z is positive and ξ is real.

Now, on account of (2.13), (2.15), (2.16) and (2.21) (cf., also, Sec. IX.3 of Ref. 1, Sec. IV.8 of Ref. 4 and Ref. 20), it is easy to check that, for $r \in (\alpha, 1)$ with α any fixed constant $0 < \alpha < 1$, the asymptotic behavior of $J_n(\nu_{k,n} r) = J_n(nz)$ comes out from the asymptotic behavior of the leading term in (2.13):

$$\left(\frac{4\xi}{1-z^2}\right)^{1/4} \frac{\mathrm{Ai}(n^{2/3}\xi)}{n^{1/3}} .$$

For r near 1, since $(\frac{4\xi}{1-z^2})^{1/4} = O(1)$ (cf. (2.15) and (2.16)), we have to analyze the behavior of the Airy function $\mathrm{Ai}(n^{2/3}\xi)$, which depends on the sign of the argument $n^{2/3}\xi$.

As a matter of fact, for ξ, z and $\nu^*(n)$ given by (3.6), (3.5) and (3.3) respectively, we have:

$$\mathrm{Ai}(n^{2/3}\xi) = \mathrm{Ai}(n^{2/3}2^{1/3}(1-(1+\nu^*(n))r)(1+O(1-(1+\nu^*(n))r)) ,$$

where we observe that the argument $n^{2/3}\xi$ is negative provided that r near 1 and $(1-(1+\nu^*(n))r) < 0$. Indeed, $(1-(1+\nu^*(n))r)$ is sufficiently small if $r = 1-O(n^{-\beta})$, where to be more precise,

$$r = 1 - O(n^{-\beta}) \text{ means } r = 1 - Kn^{-\beta} + o(n^{-\beta}) , \quad \text{for } K > 0 \quad \text{and} \quad \beta > 0 , \tag{3.7}$$

and we obtain $(1-(1+\nu^*(n))r) = O(\sup(\nu^*(n), n^{-\beta}))$.

Therefore, on account of the behavior of $\mathrm{Ai}(n^{2/3}\xi)$ for positive and negative arguments (cf. (2.22)–(2.25)), in a small neighborhood of $r = 1$ the Airy function $\mathrm{Ai}(n^{2/3}\xi)$ decays exponentially for $r < 1 - \nu^*(n) + O(\nu^*(n)^2)$ and oscillates for $1 - \nu^*(n) + O(\nu^*(n)^2) < r < 1$.

For the Bessel functions, we consider (2.13), (2.15), (2.16) and (3.4)–(3.6), we use (2.21)–(2.23), and then, by writing $J_n(nz) \equiv J_n(\nu_{k,n} r)$, we obtain

$$J_n(\nu_{k,n} r) = \left(\frac{4\xi}{1-z^2}\right)^{1/4} \frac{\mathrm{Ai}(n^{2/3}\xi)}{n^{1/3}} + O(n^{-3/2}(1-r)^{1/4}) e^{-(2/3)\sqrt{2}n(1-r)^{3/2}\alpha_n} ,$$
$$\text{for } r \ll 1 - O(n^{-2/3}) , \tag{3.8}$$

where $1 - O(n^{-2/3})$ is as in (3.7), $r \ll 1 - O(n^{-2/3})$ means $(1-r)n^{2/3} \xrightarrow{n\to\infty} \infty$ for $1 - r = o(1)$ and $\alpha_n = \alpha_n(r)$ is a well-determined sequence, $\alpha_n \xrightarrow{n\to\infty} 1$; we also obtain

$$J_n(\nu_{k,n} r) = \left(\frac{4\xi}{1-z^2}\right)^{1/4} \frac{\mathrm{Ai}(n^{2/3}\xi)}{n^{1/3}} + O(n^{-4/3}) , \quad \text{for } 1 - O(n^{-2/3}) \leq r < 1 , \tag{3.9}$$

which holds uniformly with respect to r.

Thus, considering (3.5) and (3.6), from (3.8) and (2.22)–(2.23), for $t = n^{2/3}\xi$, we obtain:

$$J_n(\nu_{k,n} r) = O(n^{-1/2+\beta/4}) e^{-(2/3)\sqrt{2}n(1-r)^{3/2}\beta_n} , \quad \text{for } r = 1 - O(n^{-\beta}) , \tag{3.10}$$

where β is a constant, $0 < \beta < \frac{2}{3}$, $\beta_n = 1 - O(n^{\beta-2/3})$, and $O(n^{-\beta})$ and $O(n^{\beta-2/3})$ are positive order functions as in (3.7).

In the same way, considering (2.13), (2.14), (3.5) and (2.22)–(2.23), we have:

$$J_n(\nu_{k,n} r) = O(n^{-1/2}) e^{-nf_n(r)} , \quad \text{for } 0 < \alpha_1 < r < \alpha_2 < 1 , \tag{3.11}$$

where α_1 and α_2 are any fixed constants $0 < \alpha_1 < \alpha_2 < 1$, the O-symbol is uniform with respect to r, and

$$0 < f_n(r) = \ln\left(\frac{1+\sqrt{1-(1+\nu^*(n))^2 r^2}}{(1+\nu^*(n))r}\right) - \sqrt{1-(1+\nu^*(n))^2 r^2} = O(1)\,.$$

Formulas (3.10) and (3.11) show that for fixed $k = 1, 2, \ldots$, and $n \to \infty$, the Bessel function $J_n(\nu_{k,n} r)$ decays exponentially, depending on the distance to $r = 1$, for $0 < r \ll 1 - O(n^{-2/3})$: we observe that, for fixed n, $f_n(r)$ is a decreasing function in $r \in (0, \frac{1}{1+\nu^*(n)})$, and, because of (3.6), $f_n(r)$ reads $\frac{2}{3}\sqrt{2}(1-r)^{3/2}\beta_n$ when $r = 1 - O(n^{-\beta})$.

We conclude that the results above for $J_n(\nu_{k,n} r)$ hold for the corresponding eigenfunctions $U_{n,k}(r,\theta)$ and $\tilde{U}_{n,k}(r,\theta)$: the thickness of the boundary layer, where they are significant and strongly oscillating functions in the arc direction, is $O(n^{-2/3})$ (cf. Fig. 1). Therefore, the proof of the existence of eigenfunctions of (1.1)–(1.3), of the whispering gallery type, associated with the eigenvalues (3.1) is complete and we have proved the following result:

Proposition 3.1. *Let* $\nu_{k,n}$ *be the zeros of* (2.3) *which have the asymptotics* (3.1). *Then, for* $\theta \in [0, 2\pi)$, *as* $n \to \infty$, *the asymptotic behavior of the corresponding eigenfunctions* $U_{n,k}(r,\theta)$ ($\tilde{U}_{n,k}(r,\theta)$, *respectively*) *is given by* (2.5) ((2.6), *respectively*) *and* (3.9)–(3.11).

Remark 3.1. It should be mentioned that the existence of whispering gallery modes, concentrating in a neighborhood of the boundary, has been obtained using different techniques (see, e.g. Secs. V and VII of Ref. 3) in connection with the Hemholtz equation in Diffraction Theory:

$$\Delta u + \frac{\omega^2}{c(x)^2} u = 0 \quad \text{in } \Omega\,,$$

where Ω is a bounded domain of $\mathbb{R}^2$, the density $c(x)$ is a smooth function in Ω and a Dirichlet (Neumann or mixed) condition on the boundary $\partial\Omega$ is prescribed.

Explicit computations of the whispering gallery eigenfunctions and the corresponding eigenvalues, based on the properties of the Bessel functions, appear in Sec. VII.2 of Ref. 3 when Ω is a circle and $c(x)$ a constant; i.e. for the Dirichlet problem (1.7) in a circle. Previous results for this problem, (1.7) in a circle, based on the examination of the eigenmodes were in Ref. 22. For more general domains, explicit computations of the eigenelements become very difficult and, in general, it is only possible to construct the so-called *quasimodes* (see Ref. 2, Sec. VII.6 of Ref. 3 and Sec. XIV.4 of Ref. 24).

On the other hand, see Ref. 15 to derive the existence of whispering gallery waves traveling on interfaces of circular domains, for a different problem from those in this paper, and also for a different technique, as well as for further references.

3.2. On the integral energy

The aim of this section is to obtain certain estimates in the L^2 norm for the whispering gallery eigenfunctions $U_{k,n}$, $\tilde{U}_{k,n}$ associated with $\nu_{k,n}$ in (3.1), and for their derivatives. These estimates will be useful in Sec. 5. They allow us to guess that the whispering gallery modes computed in this paper have most of their energy concentrated in a small neighborhood of the interface between the two components (see Remark 3.2).

For $W = U_{n,k}$ and $W = \tilde{U}_{n,k}$ in formulas (2.5) and (2.6), $\nu_{k,n}$ in (3.1), we consider polar coordinates, and we easily prove the estimates:

$$\|\nabla_y W\|_{L^2(\mathbb{R}^2-\bar{B})} \leq C_1\sqrt{n}|J_n(\nu_{k,n})| \tag{3.12}$$

and

$$\|W\|_{L^2(\mathbb{R}^2-\bar{B})} \leq C_2\frac{|J_n(\nu_{k,n})|}{\sqrt{n-1}}\,, \tag{3.13}$$

where C_1 and C_2 are constants independent of n.

Now, for $k = 1, 2, 3, \dots$, we show that the right-hand side of (3.12) and (3.13) converge towards zero, as $n \to \infty$, by proving that $J_n(\nu_{k,n}) = O_s(n^{-2/3})$.

From the recurrence relations for the Bessel functions (cf., e.g. Sec. III.13 of Ref. 26 and Sec. IX.1 of Ref. 1)

$$\nu J_n(\nu) = (n-1)J_{n-1}(\nu) - \nu J'_{n-1}(\nu)\,, \tag{3.14}$$

and the recurrence relation (3.2), when ν is a root of (2.3), $\nu = \nu_{k,n}$, we have $J_{n-1}(\nu_{k,n}) = 0$ and $J_n(\nu_{k,n}) = -J'_{n-1}(\nu_{k,n})$. Now, considering the fact that $\nu_{k,n} = \hat{\nu}_{k,n-1}$, where $\hat{\nu}_{k,n}$ has the expansion (2.19), we use (2.20), and we obtain:

$$J_n(\nu_{k,n}) = O_s(n^{-2/3})\,, \tag{3.15}$$

as previously outlined.

Besides, again using polar coordinates, for $\nu_{k,n}$ in (3.1), we prove the estimate:

$$\|\nabla_y W\|_{L^2(B)} \geq C_3 n^{1/3}\,, \tag{3.16}$$

for n sufficiently large and C_3 a constant independent of n. Indeed, (3.16) is obtained by considering

$$\|\nabla_y W\|^2_{L^2(B)} = K_1\nu^2_{k,n}\int_0^1 (J'_n(\nu_{k,n}r))^2 r\,dr + K_2 n^2\int_0^1 (J_n(\nu_{k,n}r))^2\frac{1}{r}dr\,, \tag{3.17}$$

where K_1, K_2 are constants independent of k and n. Then, we have

$$\|\nabla_y W\|^2_{L^2(B)} \geq K_2 n^2 \int_{\frac{1}{1+\nu^*(n)}}^1 (J_n(\nu_{k,n}r))^2 dr \geq K_3 n^{2/3}\,,$$

where $\nu^*(n)$ is defined in (3.3), K_3 is a constant independent of n, and the last inequality is obtained as a consequence of (3.3)–(3.5), (2.15), (2.16), expansion (3.9) which holds uniformly in r, and of the fact that $\mathrm{Ai}(t)^2$ is strictly positive in any interval which does not contain any zero of Ai. Therefore, (3.16) holds.

Remark 3.2. We observe that (3.12) and (3.15) prove that the energy integral in $\mathbb{R}^2 - \bar{B}$ of the eigenfunctions associated with the roots (3.1) is of order $O(n^{-1/6})$. Moreover, (3.16) proves that this energy is at most concentrated in B.

Now, in order to ensure that the eigenfunctions (2.5) and (2.6) associated with the eigenvalues (3.1) have their energy concentrated in a boundary layer of thickness $O(n^{-2/3})$, we have to prove that the order of magnitude of $\|\nabla_y W\|^2_{L^2(B(0,\alpha_2))}$ is smaller than the order of magnitude of $\|\nabla_y W\|^2_{L^2(B-\overline{B(0,1/(1+\nu^*(n)))})}$, for any fixed α_2 such that $0 < \alpha_2 < 1$. On account of (3.17), this fact likely comes out from the asymptotic expansion (3.11) and a similar one for the derivative $J'_n(\nu_{k,n}r)$ (cf. (2.17)), when $0 < \alpha_1 < r < \alpha_2 < 1$, and from expansions for J_n and J'_n for large orders and fixed arguments when $r\nu_{k,n} = O(1)$ (see Sec. IX.2 of Ref. 1).

Remark 3.3. We observe that bound (3.15) can also be obtained by considering (2.13) and the Taylor expansion of the Airy function in a neighborhood of the zero a_k (cf. Sec. 4) instead of the recurrence relations (3.2) and (3.14); thus, the results and bounds in this section can be extended for the more general case of roots of (2.8).

Remark 3.4. For problem (1.1)–(1.2), since Eq. (2.3) ((2.4), (2.5), (2.6), respectively) is similar to (2.8) ((2.9), (2.10), (2.11), respectively), for $n > 0$ and $R > 1$, we expect the same behavior for the corresponding eigenvalues and eigenfunctions as in the case of the unbounded domain in Sec. 3. Nevertheless, it should be noticed that the above proof of (3.1) does not apply to obtain asymptotic expansions for the zeros of (2.8), since the recurrence relations (3.2) have been used. Thus, in Sec. 4, we provide another more general proof which extend the previous one and gives the same asymptotic expansion (3.1) for the roots $\tilde{\nu}_{k,n}$ of (2.8). The behavior of the corresponding eigenfunctions is obtained from (2.12) and the results in Sec. 3.1.

4. Whispering Gallery Eigenelements of (1.1)–(1.2)

In this section we provide asymptotic expansions for the frequencies associated with the whispering gallery eigenfunctions of problem (1.1)–(1.2). Once asymptotics of the type (3.1) have been obtained for the eigenvalues $\tilde{\mu}_{k,n} = \tilde{\nu}^2_{k,n}$, the asymptotics for the eigenfunctions $V_{k,n}$, $\tilde{V}_{k,n}$ near Γ hold as in Sec. 3.1, with minor modifications: see Proposition 3.1, and formulas (2.7), (2.8), (2.10)–(2.12) and (3.8)–(3.11).

The following results state the asymptotics for the eigenelements of (1.1)–(1.2).

Lemma 4.1. *Let $\tilde{\nu}_{k,n}$ be the zeros of* (2.8). *Then, for $k = 1, 2, 3, \ldots$, the expansion*

$$\tilde{\nu}_{k,n} = n\left(1 - \frac{a_k}{2^{1/3}n^{2/3}} - \frac{1}{n} + O\left(\frac{1}{n^{4/3}}\right)\right) \tag{4.1}$$

holds, as $n \to \infty$, where a_k is the kth zero of the Airy function Ai *in* (2.18).

Proposition 4.1. *Let* $\tilde{\nu}_{k,n}$ *be the zeros of* (2.8) *which have the asymptotics* (4.1). *Then, for* $\theta \in [0, 2\pi)$, *as* $n \to \infty$, *the asymptotic behavior of the corresponding eigenfunctions* $V_{n,k}(r,\theta)$ ($\tilde{V}_{n,k}(r,\theta)$, *respectively*) *is given by* (2.10) ((2.11), *respectively*) *and* (3.9)–(3.11).

The proof of Proposition 4.1 is that of Proposition 3.1 in Sec. 3.1, with minor modifications. The proof of Lemma 4.1 is a consequence of Proposition 4.2, as we outline here below.

Since the method used to prove (3.1) for the zeros of (2.3), when $n \to \infty$, does not apply to obtain the asymptotic expansion (4.1) for the zeros of Eq. (2.8) (see Remark 3.4), in this section, we state a more general procedure to obtain the asymptotic expansions for the zeros of more general equations than (2.3) and (2.8). Then, (4.1) will be obtained as a particular case of these asymptotics.

More precisely, we look for asymptotic expansions of the roots of the equation:

$$\nu J_n'(\nu) + R_n n J_n(\nu) = 0\,, \tag{4.2}$$

where R_n is a sequence of positive numbers that converges, as $n \to \infty$, towards some constant $\tilde{R} > 0 : R_n = \tilde{R} + \beta(n)$, $\beta(n)$ being $\beta(n) = o(n^{-\gamma})$ for any γ, $\gamma > 0$ (see Remark 4.1).

As a consequence of the properties of separation of the zeros of J_n and J_n' (cf., e.g. Sec. XV of Ref. 26), Eq. (4.2) has a countable infinity of positive roots $\tilde{\nu}$, that we shall denote by $\tilde{\nu}_{k,n}$, $k = 1, 2, 3, \ldots$. We have the following result:

Proposition 4.2. *For the zeros* $\tilde{\nu}_{k,n}$ *of* (4.2), *as* $n \to \infty$, *we have the asymptotic expansions*:

$$\tilde{\nu}_{k,n} = n\left(1 - \frac{a_k}{2^{1/3} n^{2/3}} - \frac{1}{\tilde{R} n} + O\left(\frac{1}{n^{4/3}}\right)\right), \quad k = 1, 2, 3, \ldots, \tag{4.3}$$

where a_k *is the kth zero of the Airy function* Ai *in* (2.18).

Proof. To start with, we prove that, for $n \to \infty$, there are zeros of (4.2) of the form $\tilde{\nu} = n(1 + \tilde{\nu}^*(n))$ for $\tilde{\nu}^*(n) = O_s(n^{-2/3})$. In order to obtain this result, it suffices to consider two roots $\hat{\nu}_{1,n}$ and $\hat{\nu}_{2,n}$ of $J_n(\nu) = 0$ having asymptotics (2.19) for two consecutive zeros, a_1 and a_2 for example, of the Airy function; then, we take into account that $J_n'(\nu)$ changes the sign once between $\hat{\nu}_{1,n}$ and $\hat{\nu}_{2,n}$, and so does $nR_n J_n(\nu) + \nu J_n'(\nu)$; therefore, there is one zero of Eq. (4.2) between $\hat{\nu}_{1,n}$ and $\hat{\nu}_{2,n}$. This fact allows us to assert that there is

$$\tilde{\nu}^*(n) = -\frac{a(n)}{2^{1/3} n^{2/3}} + O(n^{-4/3}) \tag{4.4}$$

for a certain constant $a(n) = O_s(1)$, $a(n)$ between a_1 and a_2. In what follows, we prove that $\tilde{\nu}^*(n)$ has the asymptotic expansion:

$$\tilde{\nu}^*(n) = -\frac{a_k}{2^{1/3} n^{2/3}} - \frac{1}{\tilde{R} n} + O\left(\frac{1}{n^{4/3}}\right), \tag{4.5}$$

for a_k any fixed zero of the Airy function, $k = 1, 2, 3, \ldots$; and this gives (4.3) and (4.1) for $\tilde{R} = 1$ (see Remark 4.1). We perform the proof in three steps.

First, let us observe that to look for zeros of (4.2) of the form

$$\tilde{\nu} = n(1 + \tilde{\nu}^*(n)), \quad \tilde{\nu}^*(n) > 0, \quad \tilde{\nu}^*(n) = O(n^{-2/3}), \tag{4.6}$$

amounts to looking for $\tilde{\nu}^*(n) = O(n^{-2/3}) > 0$ satisfying:

$$n(1 + \tilde{\nu}^*(n))J_n'(n(1 + \tilde{\nu}^*(n))) + R_n n J_n(n(1 + \tilde{\nu}^*(n))) = 0. \tag{4.7}$$

Step 1: *The perturbed equation of* (4.2)
Let us admit, for the time being, that we have $\tilde{\nu}^*(n)$ satisfying (4.7). Then, we prove that

$$\xi = -\tilde{\nu}^*(n)2^{1/3} + O(\tilde{\nu}^*(n)^2) \tag{4.8}$$

satisfies

$$2^{2/3}\mathrm{Ai}(n^{2/3}\xi) - \frac{2}{R_n n^{1/3}}\mathrm{Ai}'(n^{2/3}\xi) + O\left(\frac{1}{n}\right) = 0. \tag{4.9}$$

Indeed, we take into account (2.15) and (2.16) for $z = \frac{\tilde{\nu}}{n}$, $\tilde{\nu}$ in (4.6), to obtain that ξ has the asymptotics (4.8), and

$$\left(\frac{4\xi}{1 - z^2}\right)^{1/4} = 2^{1/3} + O(\tilde{\nu}^*(n)). \tag{4.10}$$

Then, we write the expansion (2.13) for $J_n(nz)$ when $z = 1 + \tilde{\nu}^*(n)$, and (2.17) for the derivative $J_n'(nz)$, where we observe that for $\tilde{\nu}^*(n) = O(n^{-2/3})$, the argument of the Airy function $(n^{2/3}\xi)$ is of order $O(1)$. The asymptotics and estimates of the Bessel functions in terms of the Airy function and of the exponential function (2.13)–(2.21) (cf., also, Secs. IV and V of Ref. 20 and Secs. IV–VII of Ref. 21) allow us to assert that the zeros of (4.2), $\tilde{\nu} = n(1 + \tilde{\nu}^*(n))$ satisfying (4.7), verify

$$2^{2/3}R_n\mathrm{Ai}(n^{2/3}\xi) - 2n^{-1/3}\mathrm{Ai}'(n^{2/3}\xi) + O(n^{-1}) = 0, \tag{4.11}$$

which reads (4.9). Thus, for ξ in (4.8) to satisfy (4.9) is a necessary condition for $\tilde{\nu}$ in (4.6) to be a zero of (4.2).

Therefore, we proceed as follows: we look for asymptotic expansions of $\tilde{\nu}^*(n)$ and ξ satisfying (4.9) and then we prove that the corresponding expansion for $\tilde{\nu}$ in (4.6) is in fact an expansion of a true zero of (4.2).

Step 2: *The zeros of the perturbed equation* (4.9)
We look for ξ such that $n^{2/3}\xi = O(1)$ satisfies (4.9). On account of $2R_n^{-1}n^{-1/3} \to 0$, as $n \to \infty$, $n^{2/3}\xi$ is expected to be near a zero of the Airy function. Hence, we perform the change of variable: $n^{2/3}\xi = \tau + a_s$, for a_s any fixed zero of the Airy function ($\mathrm{Ai}(a_s) = 0$, $a_s < 0$) and then, we expand Ai and Ai′ in Taylor series in a neighborhood of a_s to obtain τ such that the corresponding ξ satisfies Eq. (4.9).

In fact, considering (4.9), we can write:

$$
\begin{aligned}
&2^{2/3}\mathrm{Ai}(a_s+\tau) - \frac{2}{R_n n^{1/3}}\mathrm{Ai}'(a_s+\tau) + O\left(\frac{1}{n}\right)\\
&\quad = 2^{2/3}\left(\mathrm{Ai}(a_s) + \tau\mathrm{Ai}'(a_s) + \frac{\tau^2}{2}\mathrm{Ai}''(a_s) + \frac{\tau^3}{6}\mathrm{Ai}'''(a_s) + \cdots\right)\\
&\qquad - \frac{2}{R_n n^{1/3}}\left(\mathrm{Ai}'(a_s) + \mathrm{Ai}''(a_s)\tau + \frac{\tau^2}{2}\mathrm{Ai}'''(a_s) + \cdots\right) + O\left(\frac{1}{n}\right) = 0\,.
\end{aligned}
$$

On account of (2.18) and $\mathrm{Ai}(a_s) = 0$, $\mathrm{Ai}''(a_s)$ also vanishes, and we have

$$
\left(2^{2/3}\tau - \frac{2}{R_n n^{1/3}}\right)\mathrm{Ai}'(a_s) - \left(2^{2/3}\frac{\tau}{6} - \frac{2}{R_n n^{1/3}}\frac{1}{2}\right)\tau^2\mathrm{Ai}'''(a_s) + \cdots + O(n^{-1}) = 0\,. \tag{4.12}
$$

By choosing

$$
\tau = \frac{2}{R_n n^{1/3} 2^{2/3}}\,,
$$

the term accompanying $\mathrm{Ai}'''(a_s)$ in (4.12) is already of order $O(n^{-1})$, and Eq. (4.9) is satisfied for this value of τ. Thus, we take ξ such that

$$
n^{2/3}\xi = a_s + \frac{2}{R_n n^{1/3} 2^{2/3}}\,,
$$

or equivalently, in terms of $\tilde{\nu}^*(n)$,

$$
\tilde{\nu}^*(n)(1 + O(\tilde{\nu}^*(n)^2)) = -\frac{a_s}{2^{1/3} n^{2/3}} - \frac{2}{R_n n^{1/3} 2^{2/3} 2^{1/3} n^{2/3}}\,.
$$

Then, we solve this equation in $\nu^*(n)$, i.e.

$$
K_n n^{2/3}\tilde{\nu}^*(n)^2 + n^{2/3}\tilde{\nu}^*(n) + \frac{a_s}{2^{1/3}} + \frac{1}{n^{1/3} R_n} = 0\,,
$$

where constant $K_n = K + O(\tilde{\nu}^*(n))$ converge towards some constant K different from zero, and we obtain formula (4.5) for $\tilde{\nu}^*(n)$ being $a_k \equiv a_s$. Taking into account that $R_n - \tilde{R} = O(n^{-\gamma})$ for any positive γ, the asymptotic expansion (4.5) also reads

$$
\tilde{\nu}^*(n) = -\frac{a_k}{2^{1/3} n^{2/3}} - \frac{1}{R_n n} + O\left(\frac{1}{n^{4/3}}\right).
$$

Thus, we have proved that ξ in (4.8) and $\tilde{\nu}^*(n)$ in (4.5) satisfy (4.9).

Step 3: *The true zeros of* (4.2)

Considering (4.6), we denote by ν_{per} the $\tilde{\nu}$ corresponding to the $\tilde{\nu}^*(n)$ above

$$
\nu_{\mathrm{per}} = n\left(1 - \frac{a_k}{2^{1/3} n^{2/3}} - \frac{1}{R_n n} + O\left(\frac{1}{n^{4/3}}\right)\right). \tag{4.13}
$$

In what follows, we prove that ν_{per} is in fact a small perturbation of the zeros of (4.2) and, more precisely, that there are zeros ν_{ex} of (4.2) of the form $\nu_{\mathrm{ex}} = \nu_{\mathrm{per}} + O(n^{-1/3})$; i.e. there are zeros of (4.2), which have the asymptotic expansion (4.13).

In order to perform this proof, we first observe that, on account of (4.11) (again because of (2.13)–(2.21) and (4.10)), $\nu_{\rm per}$ in (4.13) satisfies the equation:

$$\nu_{\rm per} J_n'(\nu_{\rm per}) + R_n n J_n(\nu_{\rm per}) + O(n^{-1/3}) = 0\,. \tag{4.14}$$

We denote by $\nu_{\rm ex}$ an exact zero of (4.7) near $\nu_{\rm per}$ and by $F(\nu, n)$ the function $F(\nu, n) \equiv nR_n J_n(\nu) + \nu J_n'(\nu)$. Obviously, from (4.7) and (4.14), $F(\nu_{\rm ex}, n) = 0$ and $F(\nu_{\rm per}, n) = O(n^{-1/3})$. Then, the Taylor expansion in a neighborhood of $\nu_{\rm ex}$ gives the existence of $\hat{\nu}$ between $\nu_{\rm ex}$ and $\nu_{\rm per}$ such that:

$$F(\nu_{\rm per}, n) = F(\nu_{\rm ex}, n) + \frac{dF}{d\nu}(\hat{\nu}, n)(\nu_{\rm per} - \nu_{\rm ex})\,.$$

Thus,

$$|\nu_{\rm per} - \nu_{\rm ex}| \le \left| \frac{O(n^{-1/3})}{\frac{dF}{d\nu}(\hat{\nu}, n)} \right|, \tag{4.15}$$

and we prove that $|\frac{dF}{d\nu}(\hat{\nu}, n)| \ge C$, for C a certain constant, $C > 0$, which gives the existence of the true zero of (4.2) $\nu_{\rm ex}$ at a distance less than or equal to $O(n^{-1/3})$ from $\nu_{\rm per}$. Indeed, we have

$$\frac{dF}{d\nu}(\nu, n) = nR_n J_n'(\nu) + J_n'(\nu) + \nu J_n''(\nu)\,,$$

where, on account of equation satisfied by $J_n(\nu)$,

$$J_n'(\nu) + \nu J_n''(\nu) = -\left(\nu - \frac{n^2}{\nu}\right) J_n(\nu)\,,$$

we verify

$$\left| \frac{O(n^{-1/3})}{\frac{dF}{d\nu}(\hat{\nu}, n)} \right| = \left| \frac{O(n^{-4/3})}{R_n J_n'(\hat{\nu}) - \left(\frac{\hat{\nu}}{n} - \frac{n}{\hat{\nu}}\right) J_n(\hat{\nu})} \right| \le O\left(\frac{1}{n^{1/3}}\right). \tag{4.16}$$

The inequality in (4.16) is obtained from the fact that $\hat{\nu}$ is already of the form $\hat{\nu} = n(1 + \hat{\nu}^*(n))$ for some small $\hat{\nu}^*(n) = O(n^{-2/3})$ (see (4.4), (4.6) and (4.13)); then, (2.13) and (2.17) hold for $\xi n^{2/3}$ of order $O(1)$ and the dominant term in the denominator of the left-hand side of (4.16) is of the order greater than or equal to $O(n^{-1})$.

Inequalities (4.15), (4.13) and (4.16) ensure:

$$\nu_{\rm ex} = n\left(1 - \frac{a_k}{2^{1/3} n^{2/3}} - \frac{1}{R_n n} + O\left(\frac{1}{n^{4/3}}\right)\right), \tag{4.17}$$

as outlined previously, and the proposition is proved. □

Remark 4.1. In order to obtain (3.1) and (4.1) from Proposition 4.2, it is worthy observing that Eq. (4.2) is (2.3) when $R_n = 1$ and (2.8) when $R_n = \frac{R^{2n}+1}{R^{2n}-1}$. In this case, R_n converge towards 1 as $n \to \infty$, and, $R_n = 1 + o(n^{-\gamma})$ for any positive γ, since

$$R_n = \frac{R^{2n}+1}{R^{2n}-1} = 1 + \frac{2}{R^{2n}} + \frac{2}{R^{4n}} + \cdots . \quad (4.18)$$

We observe that in the case where $R_n = 1$, (4.17) coincides with (3.1), and the proof of (4.17) extends that performed to obtain (3.1).

Remark 4.2. It should be mentioned that (3.1) (and (4.1), (4.3) or (4.17)) differs from the asymptotics for the zeros of the Bessel functions (2.19), obtained in Refs. 3 and 21, in the terms of order $O(1)$.

We also note that, in fact, formula (4.4) already ensures the existence of eigenfunctions of the whispering gallery type, but (4.1) provides asymptotic expansions for the corresponding eigenvalues up to the order $O(1)$.

5. On a Vibrating Membrane with One Single Concentrated Mass

We address the asymptotic behavior of the eigenfunctions associated with certain high frequencies λ^ε of (1.4), as $\varepsilon \to 0$. Throughout this section we consider $\lambda^\varepsilon = \lambda^\varepsilon_{i(\varepsilon)} = O(1)$, λ^ε converging towards some positive λ^*.

The asymptotic expansions in Ref. 5 show that the eigenfunctions associated with the high frequencies of (1.1)–(1.3), $\lambda = \frac{\lambda^*}{\varepsilon^{m-2}}$ for small ε, provide correcting terms for certain eigenfunctions associated with the high frequencies of (1.4). Using this idea and the whispering gallery eigenfunctions in Sec. 3.1, in this section, we prove that there are eigenfunctions of (1.4) associated with $\lambda^\varepsilon = O(1)$ which concentrate in a neighborhood of Γ^ε, and almost vanish outside, as stated in Theorem 5.1. The results in this section extend and improve those in Sec. VI of Ref. 5.

It is known[5] that the eigenfunctions u^ε of (1.4), u^ε associated with λ^ε, converge weakly in $H^1_0(\Omega)$ either towards $u^0 \neq 0$, u^0 being an eigenfunction of (1.7) corresponding with the eigenvalue $\lambda^* = \lambda^0$, or towards 0 when λ^* is not an eigenvalue of (1.7). In addition, it has been proved that certain of these eigenfunctions u^ε are strongly oscillating functions inside εB and, for particular subsequences $\{\varepsilon_k\}_{k=1}^\infty$, some correcting terms which improve the convergence of u^ε in $H^1_0(\Omega)$ have been provided.[5] More precisely, the correcting terms have been constructed in the case where $\lambda^* = \lambda^0$ is an eigenvalue of (1.7), with the corresponding eigenfunction u^0 satisfying $u^0(0) \neq 0$, and in the case where λ^* is not an eigenvalue of (1.7). These correctors allow us to assert that the structure of u^ε is deeply involved with the sequence ε_k considered (see Remark 6.2 of Ref. 5 and Remark 5.4 below for the case where $u^0(0) = 0$).

The following theorem also provides the sequence ε in order to obtain whispering gallery modes of (1.4) concentrating on the interface Γ^ε (see Remarks 5.1 and 5.2).

Theorem 5.1. *Let λ^* be any positive number, λ^* not eigenvalue of* (1.7). *For fixed $k = 1, 2, \ldots$, let ε be ranging in the sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ defined by*:

$$\varepsilon_n = \left(\frac{\lambda^*}{\mu_{k,n}}\right)^{\frac{1}{m-2}}, \tag{5.1}$$

where $\mu_{k,n} = \nu_{k,n}^2$, $\nu_{k,n}$ defined in (3.1). *Then, there is a sequence δ^{ε_n}, $\delta^{\varepsilon_n} \to 0$ as $n \to \infty$, such that the interval $[\lambda^* - \delta^{\varepsilon_n}, \lambda^* + \delta^{\varepsilon_n}]$ contains eigenvalues of* (1.4). *Moreover, there is $u^{\varepsilon_n}, u^{\varepsilon_n}$ with norm 1 in $H_0^1(\Omega)$, u^{ε_n} belonging to the eigenspace associated with all the eigenvalues λ^{ε_n} in $[\lambda^* - \delta^{\varepsilon_n}, \lambda^* + \delta^{\varepsilon_n}]$ such that*:

$$\|u^{\varepsilon_n} - \alpha^{\varepsilon_n}(\mathcal{T}_x W_n \psi)\|_{H_0^1(\Omega)} \leq C(k)\varepsilon_n^{\alpha_0}, \tag{5.2}$$

where α_0 is a fixed constant $\alpha_0 \in (0,1)$, $C(k)$ is a constant independent of ε_n, α^{ε_n} is the constant $\alpha^{\varepsilon_n} = 1/\|(\mathcal{T}_x W_n \psi)\|_{H_0^1(\Omega)}$, $W_n(y)$ is the solution of (2.1), *$W_n(y) = U_{k,n}(y)$ or $W_n(y) = \tilde{U}_{k,n}(y)$, y is the variable $y = \frac{x}{\varepsilon_n}$ (the local variable), $\mathcal{T}_x W_n$ is the function $\mathcal{T}_x W_n(x) = W_n(\frac{x}{\varepsilon_n})$, and ψ is any smooth function taking value 1 for $|x| < R_1$ and 0 for $|x| > R_2$, $0 \leq \psi \leq 1$, R_1 and R_2 being two fixed constants such that $R_1 < R_2$ and $B(0, R_2) \subset \Omega$.*

Proof. By performing computations in polar coordinates, the fact that, for fixed k, $\nu_{k,n}$ is given by (3.1), $\nu_{k,n} \to \infty$ as $n \to \infty$, and formulas (3.12), (3.13) and (3.16), lead us to obtain the following estimates:

$$\|\nabla_y W_n\|_{L^2(B(0,\frac{R_2}{\varepsilon_n}) - \overline{B(0,\frac{R_1}{\varepsilon_n})})} \leq C_1 \varepsilon_n^n \frac{\sqrt{n}}{R_1^n} |J_n(\nu_{k,n})|, \tag{5.3}$$

$$\|W_n\|_{L^2(B(0,\frac{R_2}{\varepsilon_n}) - \bar{B})} \leq C_2 \frac{|J_n(\nu_{k,n})|}{\sqrt{n}} \tag{5.4}$$

and

$$\|\nabla_y W_n\|_{L^2(B)} \geq C_3, \tag{5.5}$$

where C_i, $i = 1, 2, 3$, are constants independent of n. Moreover, on account of (3.15), we can assert that the constants depending on n accompanying C_1 and C_2, on the right-hand side of (5.3) and (5.4) respectively, converge towards zero as $n \to \infty$.

Then, we prove estimate (5.2) by rewriting the proof of Theorem 6.2 of Ref. 5, with minor modifications, for $\delta^{\varepsilon_n} = \varepsilon_n^{1-\alpha_0}$. We outline here the proof.

On account of $\|\nabla_x u\|_{L^2(\Omega)} = \|\nabla_y u\|_{L^2(\varepsilon_n^{-1}\Omega)}$, we perform the calculations in the local variable $y = \frac{x}{\varepsilon_n}$.

Let $\mathbf{A}^\varepsilon$ be the positive, compact and symmetric operator on $H_0^1(\varepsilon^{-1}\Omega)$ defined by:

$$\langle \mathbf{A}^\varepsilon U, V \rangle_{H_0^1(\varepsilon^{-1}\Omega)} = \frac{1}{\varepsilon^{m-2}} \int_B UV\, dy + \varepsilon^2 \int_{\varepsilon^{-1}\Omega - \bar{B}} UV\, dy, \quad \forall U, V \in H_0^1(\varepsilon^{-1}\Omega).$$

Making the change of variable from x to y in (1.4), we obtain that the eigenvalues of $\mathbf{A}^\varepsilon$ are $1/\lambda^\varepsilon$.

Let us consider $W^{\varepsilon_n}(y) = W_n(y)\psi^{\varepsilon_n}(y)$, with $W_n = U_{k,n}(y)$ defined in (2.5) and $\psi^{\varepsilon_n}(y) = \psi(\varepsilon_n y)$ (nothing changes if W_n is defined in (2.6)), W_n and ψ as in the statement of the theorem.

Let us define $\tilde{W}^{\varepsilon_n} = \frac{W^{\varepsilon_n}}{\|W^{\varepsilon_n}\|_{H_0^1(\varepsilon_n^{-1}\Omega)}}$. Let us admit, for the time being, that

$$\left\| \mathbf{A}^{\varepsilon_n}\tilde{W}^{\varepsilon_n} - \frac{1}{\lambda^*}\tilde{W}^{\varepsilon_n} \right\|_{H_0^1(\varepsilon_n^{-1}\Omega)} \leq C(k)\beta^{\varepsilon_n}\,, \tag{5.6}$$

where $\beta^{\varepsilon_n} = \varepsilon_n|J_n(\nu_{k,n})|$. Then, we apply Lemma 1.1 in Sec. III.1 of Ref. 19 to the operator $A = \mathbf{A}^{\varepsilon_n}$ and the Hilbert space $H = H_0^1(\varepsilon_n^{-1}\Omega)$, and we obtain the result in the statement of the theorem.

In order to obtain (5.6), we prove:

$$\|W^{\varepsilon_n}\|_{H_0^1(\varepsilon_n^{-1}\Omega)} \geq C_4(k) \tag{5.7}$$

and

$$\begin{aligned} &\left| \left\langle \mathbf{A}^{\varepsilon_n}W^{\varepsilon_n} - \frac{1}{\lambda^*}W^{\varepsilon_n}, V \right\rangle_{H_0^1(\varepsilon_n^{-1}\Omega)} \right| \\ &\qquad \leq C_5(k)\beta^{\varepsilon_n}\|V\|_{H_0^1(\varepsilon_n^{-1}\Omega)}\,, \quad \forall V \in H_0^1(\varepsilon_n^{-1}\Omega)\,, \end{aligned} \tag{5.8}$$

for some constants $C_4(k), C_5(k)$ independent of ε_n.

Formula (5.7) is a consequence of the definition of W^{ε_n}, which takes the value (2.5) in B, and of (5.5).

In relation to (5.8), the definitions of $\mathbf{A}^{\varepsilon_n}$ and W^{ε_n} allow us to write:

$$\begin{aligned} \left\langle \mathbf{A}^{\varepsilon_n}W^{\varepsilon_n} - \frac{1}{\lambda^*}W^{\varepsilon_n}, V \right\rangle_{H_0^1(\varepsilon_n^{-1}\Omega)} &= \frac{1}{\varepsilon_n^{m-2}}\int_B W_n V\,dy - \frac{1}{\lambda^*}\int_{\mathbb{R}^2} \nabla_y W_n \cdot \nabla_y V\,dy \\ &+ \frac{1}{\lambda^*}\int_{\mathbb{R}^2 - \overline{B(0,\frac{R_1}{\varepsilon_n})}} \nabla_y W_n \cdot \nabla_y V\,dy - \frac{1}{\lambda^*}\int_{B(0,\frac{R_2}{\varepsilon_n}) - \overline{B(0,\frac{R_1}{\varepsilon_n})}} \nabla_y (W_n\psi^{\varepsilon_n}) \cdot \nabla_y V\,dy \\ &+ \varepsilon_n^2 \int_{B(0,\frac{R_1}{\varepsilon_n}) - \bar{B}} W_n V\,dy + \varepsilon_n^2 \int_{B(0,\frac{R_2}{\varepsilon_n}) - \overline{B(0,\frac{R_1}{\varepsilon_n})}} W_n \psi^{\varepsilon_n} V\,dy\,, \end{aligned}$$

for any $V \in H_0^1(\varepsilon_n^{-1}\Omega)$. We take into account that W_n satisfies (2.1) with $K = 0$; so that the variational formulation of (1.1)–(1.3) for $\lambda = \frac{\lambda^*}{\varepsilon_n^{m-2}}$ leads us to cancel the first two integrals. For the other integrals, we apply the Schwarz and Poincaré inequalities, we take into account the boundedness of ψ^{ε_n} and its derivatives, and relations (5.3) and (5.4), and then we obtain (5.8). Therefore, the theorem is proved. □

Remark 5.1. It should be mentioned that the result in Theorem 5.1 here, along with results in Theorem 4.1 and Propositions 4.1–4.3 of Ref. 5, allow us to assert that $\alpha^{\varepsilon_n}(\mathcal{T}_x W_n\psi)$ behaves as a corrector for certain eigenfunctions u^{ε_n} which converge towards zero in $H_0^1(\Omega)$-weak. On the other hand, the above-mentioned results also allow us to show the approach to these eigenfunctions u^{ε_n} through functions which concentrate their support in the neighborhood of Γ^{ε_n} of width $O(n^{-2/3}\varepsilon_n)$. Thus, on account of (5.1), the thickness of the boundary layer is $O(n^{-\frac{2(m+1)}{3(m-2)}})$ in terms of n, and $O(\varepsilon_n^{\frac{m+1}{3}})$ in terms of $\varepsilon = \varepsilon_n$.

Remark 5.2. Since we do not know the distance between two consecutive eigenvalues of (1.4), it is clear that when justifying computations in Theorem 5.1 we cannot ensure that $\alpha^{\varepsilon_n}(\mathcal{T}_x W_n \psi)$ approaches only one eigenfunction u^{ε_n}. This fact has already been pointed out in Ref. 5 (see also Ref. 13, for a different problem). Besides, we have obtained quasimodes of (1.4) instead of true modes. True modes for (1.4) are likely to be obtained, in the case where both Ω and B are circles, using the technique in Secs. 2–4.

On the other hand, it is clear that the technique used to prove Theorem 5.1 can also be applied to different problems: let us mention, for example, the case of vibrating plates with concentrated masses (cf. Ref. 6), the case where the mass εB is of order $O(1)$, i.e. εB is replaced by Ω_2, or the case where this part Ω_2 is very stiff instead of very heavy (cf. Ref. 13).

Remark 5.3. We observe that in the case where Ω is a circle, using the results in Sec. VII.2 of Ref. 3 for problem (1.7) and the technique to prove Theorem 5.1 (see also Ref. 5), it is also possible to find whispering gallery eigenfunctions of (1.4) concentrating in a thin layer near $\partial\Omega$. Indeed, since the eigenelements of (1.7) approach certain eigenelements of (1.4) associated with the high frequencies $\lambda^\varepsilon = O(1)$, and the whispering gallery eigenfunctions of (1.7) are associated with very large eigenvalues of (1.7), the whispering gallery eigenfunctions of (1.4) concentrating in a neighborhood of the boundary of Ω are likely to be associated with frequencies higher than $\lambda^\varepsilon = O(1)$.

Remark 5.4. From Remark 6.2 of Ref. 5, it is clear that when (λ^0, u^0) is an eigenelement of (1.7) and $u^0(0) = 0$, and $\lambda^\varepsilon \xrightarrow{\varepsilon\to 0} \lambda^0$, the approach of u^ε through 0 inside εB is as good as the approach through an oscillating function. As a matter of fact, it should be noted that in order to improve this convergence we have the problem of matching a smooth function with a very strongly oscillating function.

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