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Spectral convergence for vibrating systems containing a part with negligible mass

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SUMMARY

We consider a set of Neumann (mixed, respectively) eigenvalue problems for the Laplace operator. Each problem is posed in a bounded domain Ω_R of $\mathbb{R}^n$, with $n = 2, 3$, which contains a fixed bounded domain B where the density takes the value 1 and 0 outside. Ω_R has a diameter depending on a parameter R, with $R \geqslant 1$, $\mathrm{diam}(\Omega_R) \to \infty$ as $R \to \infty$ and the union of these sets is the whole space $\mathbb{R}^n$ (the half space $\{x \in \mathbb{R}^n / x_n < 0\}$, respectively). Depending on the dimension of the space n, and on the boundary conditions, we describe the asymptotic behaviour of the eigenelements as $R \to \infty$. We apply these asymptotics in order to derive important spectral properties for vibrating systems with concentrated masses. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: spectral analysis; concentrated masses; convergence of operators

1. INTRODUCTION

This paper deals with the approach to spectral problems for the Laplace operator in unbounded domains via problems in bounded domains. These kinds of spectral problems can appear in different fields of mechanics. This is the case, for instance, of the study of local behaviours of eigenmodes for vibrating systems containing either parts with negligible mass or concentrated masses. We also apply the approach in this paper to obtain certain remarkable spectral properties for vibrating systems with concentrated masses.

As is well known, the study of the vibrations of certain mechanical systems containing a part with negligible small mass leads us to the study of the spectral problem: Find λ and

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$u \neq 0$ such that

$$\begin{cases} -\Delta u = \lambda u \quad \text{in } B \\ -\Delta u = 0 \quad \text{in } \omega \\ [u] = \left[\dfrac{\partial u}{\partial n}\right] = 0 \quad \text{on } \Gamma \end{cases} \tag{1}$$

and

$$u = 0 \quad \text{on } \Gamma_\omega \tag{2}$$

or

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_\omega \tag{3}$$

Here, B and ω are two open bounded domains of $\mathbb{R}^n$, with $n = 2, 3$. B and ω have smooth boundaries, Γ is the boundary of B, $\Gamma \cup \Gamma_\omega$ is the boundary of ω, $\bar{n}$ is the unit outward normal to Γ (Γ_ω, respectively), $\bar{\Gamma} \cap \bar{\Gamma}_\omega = \emptyset$, and the brackets denote the jump across Γ of the enclosed quantities. We have assumed that the part of the system with negligible mass fills the outer domain ω: the density takes the value 1 in B and 0 in ω; Γ is the *interface* between the two domains B and ω.

In the case where ω is the outer domain of B, that is, $\omega = \mathbb{R}^n - \bar{B}$, condition (2) on Γ_ω becomes

$$u(x) \to c \text{ as } |x| \to \infty \text{ when } n = 2, \text{ and, } u(x) \to 0 \text{ as } |x| \to \infty \text{ when } n = 3 \tag{4}$$

where c is an unknown but well determined constant.

As a matter of fact, problems (1), (2) and (1), (3) for $n = 2$ (for $n = 3$, respectively) appear, for instance, in a natural way when studying vibrating membranes (bodies, respectively) containing a very heavy inclusion. On the other hand, problem (1), (4) appears as a *microscopic* or *local problem* when studying, for example, vibrating membranes (bodies) with concentrated masses: that is, the vibrations of systems composed of a membrane (body) which contains very heavy small inclusions, *the concentrated masses* (cf. problem (56)). More specifically, (1), (4) is clearly involved with the low frequency vibrations for vibrating systems with concentrated masses inside the domain. It is the so-called *local problem* for these vibrating systems which have been studied by many authors over the last few decades: let us mention, for instance, References [1–3], Section III.5 in Reference [4] and Section VII.10 in Reference [5], and Reference [6] for an extensive bibliography on the subject.

We refer to Sections IV.6 and IV.8 in Reference [5] for the variational formulation of problems (1),(2), (1),(3) and (1),(4) in $H^1(B)$. All these problems are standard eigenvalue problems with a discrete spectrum

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \cdots \xrightarrow{n \to \infty} \infty$$

where the classical convention of repeated eigenvalues is adopted; the corresponding eigenfunctions form a basis of $L^2(B)$ and of $H^1(B)$. Obviously, the value 0 is the first eigenvalue λ_1 of (1), (3) for $n = 2, 3$ and of (1), (4) only for $n = 2$.

In fact, using the technique in Section VII.11 of Reference [5] and Section III.1 of Reference [4], it can be proved that when the diameter of ω converges towards ∞, the limit of the spectrum of problem (1), (2) is the spectrum of (1), (4). Also, the convergence of the eigenfunctions holds in the suitable Hilbert spaces. We refer to Reference [7] in connection with the structure of certain eigenfunctions associated with the high frequencies of (1), (2) and (1), (4).

In contrast, as far as we know, no mention is made in the literature on what would be the corresponding condition (3) when ω is the outer domain $\mathbb{R}^n - \bar{B}$, and consequently, neither the behaviour for the limit of the spectrum of (1), (3), when the diameter of ω converges towards ∞, is known. In this paper (cf. Sections 2–4), we describe the limit behaviour of the eigenelements of (1), (3) as $\text{diam}(\omega) \to \infty$. We also consider a different geometry of the domains ω and B, which involves the lower half space $\mathbb{R}^{n-} = \{x \in \mathbb{R}^n / x_n < 0\}$, and by prescribing mixed boundary conditions on $\{x_n = 0\}$, we analyze the limit for the spectrum of the new problems and that of the associated eigenfunctions (see Section 5).

We show that the limit problem of (1), (3), as $\text{diam}(\omega) \to \infty$, is (1), (4) for the dimension of the space $n = 2$, while the limit problem is another different eigenvalue problem for the dimension $n = 3$, namely, the equation in B in (1), (4), must be replaced by

$$-\Delta u = \lambda \left(u - \frac{1}{|B|} \int_B u \, \mathrm{d}x \right) \quad \text{in } B \tag{5}$$

(see (13) and Remark 6.4). We give convergence results on the eigenvalues and the corresponding eigenfunctions in the way stated by Theorems 2.1–4.2.

In Section 2 we outline the different problems arising throughout the paper and introduce some notations and certain known auxiliary results from spectral perturbation theory that will be used in the rest of the paper. In Section 2.1 we provide certain results of comparison for the spectrum of (1), (3) and bounds as $\text{diam}(\omega) \to \infty$. In Sections 3 and 4 we give convergence results and proofs for the case where $n = 2$ and 3, respectively.

The technique and results in Sections 2.1 and 3 extend to the case where B and ω are subdomains of the lower half space $\mathbb{R}^{n-}$ with a part of the boundary in contact with $\{x_n = 0\}$, the boundary conditions being a Dirichlet or Neumann condition on $\partial B \cap \{x_n = 0\}$ and a Neumann one outside. In the case where a Dirichlet condition is imposed on $\partial B \cap \{x_n = 0\}$ the eigenvalues are strictly positive numbers, and the convergence of the spectrum for these mixed problems, as $\text{diam}(\omega) \to \infty$, is proved in Section 5.

It is also worth mentioning that the technique can be applied to the study of other spectral problems for different elliptic operators posed in domains with different geometries such as those appearing in vibrating elastic structures with corners. In this framework, we observe that the interest in this kind of approaches in practical applications has been indicated in Reference [8] for the case of stationary elliptic problems in certain unbounded domains with Neumann conditions on the boundary (see also Section VIII in Reference [9]).

In our case, the above-mentioned mixed problems in $\mathbb{R}^{n-}$ are local problems for vibrating systems with concentrated masses near the boundary. In Section 6 we provide a sample which illustrates the interest of the results in Sections 5 in order to describe the asymptotic behaviour of the spectrum of an eigenvalue problem associated with a vibrating system with many concentrated masses. Let us mention References [6,10–14] in connection with these vibrating systems (see also Reference [6] for more references).

It should be noticed that an important fact which gives rise to this different asymptotic behaviour of the spectrum of (1), (3) for the dimensions $n=2$ and 3 of the space is that harmonic functions with a bounded energy in an unbounded domain can converge towards a constant at the infinity. Consequently, they belong to the functional space where the eigenvalue problem (1), (4) is posed only for $n=2$ (cf. Theorems 2.1 and 4.2 and Remarks 2.1 and 4.2).

Also, we note that this different asymptotic behaviour of the spectrum of (1), (3) for both dimensions allows us to prove a different behaviour of the spectrum of vibrating systems with many concentrated masses for $n=2$ and 3. Additionally, we show that our results are sharp (cf. Remarks 5.2 and 6.2). In particular, we make clear that the fact that the low frequencies for vibrating systems with concentrated masses give rise only to local vibrations of the masses is related with the fact that the first eigenvalues of problem (1), (2) and problem (1), (3) converge, as $\operatorname{diam}(\omega)\to\infty$, towards the same value which is the first eigenvalue of (1), (4) (cf. Remarks 5.2, 6.1 and 6.5).

Finally, we emphasize that the limit spectra obtained in this paper are very different depending on the geometry of the domains, on the dimension of the space and on the parameter-dependent problems. The minimax principle, extension operators and certain results on spectral perturbation theory and potential theory prove to be important tools throughout the paper. Nevertheless, it should also be pointed out that the results and techniques in this paper are different from those in the literature on vibrating systems.

2. STATEMENT OF THE PROBLEMS AND PRELIMINARIES

Let us consider B and Ω_1 two open bounded domains of $\mathbb{R}^n$, $n=2,3$, such that $\bar{B}\subset\Omega_1$. Let Ω_R be a sequence of domains such that, $\bar{B}\subset\Omega_R$, $\bar{\Omega}_1\subset\Omega_R$ and $\bar{\Omega}_R\subset\Omega_{R'}$ for $1<R<R'$. In addition, we assume that $\operatorname{diam}(\Omega_R)\to\infty$ as $R\to\infty$ and, for sufficiently large R, Ω_R contains any fixed domain of $\mathbb{R}^n$. We assume that B and Ω_R have smooth boundaries; Γ is the boundary of B and Γ_R is the boundary of Ω_R for $R\geqslant 1$. For instance, this situation holds in the particular case where Ω_R is homothetic of Ω_1 of radius R, i.e. $\Omega_R=R\Omega_1$, but more general situations can also hold (cf. Remark 4.3).

For fixed $R\geqslant 1$, we consider the eigenvalue problem (1), (3) where we set $\omega\equiv\Omega_R-\bar{B}$. That is, the problem

$$\begin{cases} -\Delta u=\lambda u \quad \text{in } B \\ -\Delta u=0 \quad \text{in } \Omega_R-\bar{B} \\ [u]=\left[\dfrac{\partial u}{\partial n}\right]=0 \quad \text{on } \Gamma \\ \dfrac{\partial u}{\partial n_R}=0 \quad \text{on } \Gamma_R \end{cases} \tag{6}$$

where $\bar{n}$ ($\bar{n}_R$, respectively) is the unit outward normal to Γ (Γ_R, respectively).

As is well known, (6) has the variational formulation: Find λ and $u \neq 0$, $u \in H^1(\Omega_R)$, such that

$$\int_{\Omega_R} \nabla u \cdot \nabla v \, dx + \int_B uv \, dx = (\lambda + 1) \int_B uv \, dx, \quad \forall v \in H^1(\Omega_R) \tag{7}$$

For fixed R, let $\{\lambda_i^R\}_{i=1}^{\infty}$ be the sequence of eigenvalues of (6), converging to ∞ as $i \to \infty$, with the classical convention of repeated eigenvalues, and let $\{u_i^R\}_{i=1}^{\infty}$ be the corresponding eigenfunctions which are assumed to be a basis of $H^1(\Omega_R)$. Obviously, $\lambda_1^R = 0$ and the corresponding eigenfunctions are the constants. The rest of eigenfunctions are a basis in the functional space

$$\mathscr{W}_R = \left\{ u \in H^1(\Omega_R) \Big/ \int_B u \, dx = 0 \right\} \tag{8}$$

they are orthogonal in $L^2(B)$, and we assume that they satisfy the normalization condition:

$$\|\nabla u_i^R\|^2_{(L^2(\Omega_R))^n} + \|u_i^R\|^2_{L^2(B)} = 1 \tag{9}$$

Let us denote by $\mathscr{V}_R$ the space $H^1(\Omega_R)$ equipped with the norm

$$\|\nabla u\|^2_{(L^2(\Omega_R))^n} + \|u\|^2_{L^2(B)} \tag{10}$$

equivalent to the usual norm in $H^1(\Omega_R)$, and by $\mathscr{W}_R$ the subspace (8) of $\mathscr{V}_R$.

On the other hand, let us consider the eigenvalue problem (1), (4) for $\omega = \mathbb{R}^n - \bar{B}$, which reads

$$\begin{cases} -\Delta_y u = \lambda u \quad \text{in } B \\ -\Delta u = 0 \quad \text{in } \mathbb{R}^n - \bar{B} \\ [u] = \left[\dfrac{\partial u}{\partial n}\right] = 0 \quad \text{on } \Gamma \\ u(x) \to c, \text{ as } |x| \to \infty \quad \text{when } n = 2 \\ u(x) \to 0, \text{ as } |x| \to \infty \quad \text{when } n = 3 \end{cases} \tag{11}$$

It has the variational formulation: Find λ, $u \in \mathscr{V}$, $u \neq 0$, such that:

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx + \int_B uv \, dx = (\lambda + 1) \int_B uv \, dx, \quad \forall v \in \mathscr{V} \tag{12}$$

where $\mathscr{V}$ is the space completion of $\mathscr{D}(\mathbb{R}^n)$ for the norm $\|\nabla u\|_{(L^2(\mathbb{R}^n))^n} + \|u\|_{L^2(B)}$.

As is well known (cf., for instance, Section III.5 of Reference [4] and Section IV.8 of Reference [5]), on account of the dense and compact imbedding $\mathscr{V} \subset L^2(B)$, problem (12) has a discrete spectrum. Let $\{\lambda_i^0\}_{i=1}^{\infty}$ be the sequence of eigenvalues with the convention of repeated eigenvalues; the corresponding eigenfunctions $\{U_i^0\}_{i=1}^{\infty}$ are an orthogonal basis in $\mathscr{V}$ and in $L^2(B)$. For $n = 2$, $\lambda_1^0 = 0$ and the associated eigenfunctions are the constants; the rest of the eigenfunctions belong to space $\mathscr{W}_R$ in (8) for all $R \geqslant 1$.

In the case where $n=3$, we also consider the problem

$$\begin{cases} -\Delta U = \lambda\left(U - \dfrac{1}{|B|}\displaystyle\int_B U\,\mathrm{d}x\right) & \text{in } B \\ -\Delta U = 0 \quad \text{in } \mathbb{R}^3 - \bar{B} \\ \quad [U] = \left[\dfrac{\partial U}{\partial n}\right] = 0 \quad \text{on } \Gamma \\ U(x)\to 0, \quad \text{as } |x|\to\infty \end{cases} \tag{13}$$

We verify that problem (13) has a discrete real non-negative spectrum $\{\Lambda_i^*\}_{i=1}^{\infty}$, with the convention of repeated eigenvalues. The set of eigenfunctions $\{U_i^*\}_{i=1}^{\infty}$ is orthogonal in $\hat{\mathscr{V}}$, where $\hat{\mathscr{V}}$ is the space completion of $\mathscr{D}(\mathbb{R}^3)$ for the norm $\|\nabla u\|_{(L^2(\mathbb{R}^3))^3}$ (cf., for instance, Section I.1 in Reference [15]). As is known, $\hat{\mathscr{V}}\subset L^2(B)$ with a dense and compact imbedding.

Indeed, (13) can be written as an eigenvalue problem for a non-negative, self-adjoint, compact operator $\mathscr{A}$ in the space $\hat{\mathscr{V}}$ as follows: Find μ ($\mu=1/\lambda$) and $U\in\hat{\mathscr{V}}$, $U\neq 0$ such that $\mathscr{A}U=\mu U$, where $\mathscr{A}$ is the operator defined by

$$\langle \mathscr{A}U, V\rangle = \int_B UV\,\mathrm{d}x - \frac{1}{|B|}\int_B U\,\mathrm{d}x\int_B V\,\mathrm{d}x, \quad \forall U, V\in\hat{\mathscr{V}}$$

$\mu=0$ is an eigenvalue of $\mathscr{A}$ and the associated eigenspace is $\mathrm{Ker}(\mathscr{A})=\{U\in\hat{\mathscr{V}}/U$ is a constant in $B\}$. As a matter of fact, for $V\in\hat{\mathscr{V}}$, $\mathscr{A}V$ is an eigenelement of the dual space $\hat{\mathscr{V}}'$ that we identify with $\hat{\mathscr{V}}$ by the Riesz Theorem. Therefore, the integral formulation of (13) reads: Find λ, $U\in\hat{\mathscr{V}}$, $U\neq 0$, such that

$$\int_{\mathbb{R}^n} \nabla U\cdot\nabla V\,\mathrm{d}x = \lambda\left(\int_B UV\,\mathrm{d}x - \frac{1}{|B|}\int_B U\,\mathrm{d}x\int_B V\,\mathrm{d}x\right), \quad \forall V\in\hat{\mathscr{V}} \tag{14}$$

In Section 2.1 we obtain a first relation between the eigenvalues of (6) and those of (11) which proves to be stronger for the dimension of the space $n=2$ as we show in Section 3. In contrast, for $n=3$, in Section 4 we prove that the limit eigenvalue problem of (6) is (13).

For the sake of completeness, we introduce here two known results of spectral perturbation theory which prove to be useful in Sections 3–5. The first one provides spectral convergence for positive, self-adjoint and compact operators defined on Hilbert spaces depending on a parameter. The second one is a weaker result on the approach to the spectrum of a positive, self-adjoint and compact operator, but operating under weaker hypothesis too. We refer to Section III.1 in Reference [4] for their proofs.

Lemma 2.1

Let H_R and H_0 be two separable Hilbert spaces with the scalar products $(\cdot,\cdot)_R$ and $(\cdot,\cdot)_0$, respectively. Let $A_R\in\mathscr{L}(H_R)$ and $A_0\in\mathscr{L}(H_0)$. Let V be a linear subspace of H_0 such that $\{v/v=A_0u \text{ with } u\in H_0\}\subset V$. We assume that the following properties are satisfied:

- (C1) There exists $\mathscr{R}_R$ a linear operator $\mathscr{R}_R\in\mathscr{L}(H_0,H_R)$ such that $(\mathscr{R}_R f,\mathscr{R}_R f)_{H_R}$ converge towards $\gamma_0(f,f)_{H_0}$, as $R\to\infty$, for all $f\in V$ and a certain positive constant γ_0.
- (C2) The operators A_R and A_0 are positive, compact and self-adjoint. Moreover, $\|A_R\|_{\mathscr{L}(H_R)}$ are bounded by a constant independent of R.

(C3) For any $f \in V$, $\|A_R \mathscr{R}_R f - \mathscr{R}_R A_0 f\|_{H_R} \to 0$ as $R \to \infty$.
(C4) The family of operators A_R is uniformly compact, i.e. for any sequence $\{f_R\}_{R>1}$, $f_R \in H_R$, such that $\sup_R \|f_R\|_{H_R}$ is bounded by a constant independent of R, we can extract a subsequence $f_{R'}$ verifying

$$\|A_{R'} f_{R'} - R_{R'} v^0\|_{H_{R'}} \to 0$$

as $R' \to \infty$, for a certain $v^0 \in H_0$.

Let $\{\mu_i^R\}_{i=1}^{\infty}$ and $\{\mu_i^0\}_{i=1}^{\infty}$ be the sequences of the eigenvalues of A_R and A_0, respectively, with the classical convention of repeated eigenvalues. Let $\{w_i^R\}_{i=1}^{\infty}$ ($\{w_i^0\}_{i=1}^{\infty}$, respectively) be the corresponding eigenfunctions in H_R, which are assumed to be orthonormal (H_0, respectively). Then, for each $k = 1, 2, \ldots, \mu_k^R \to \mu_k^0$ as $R \to \infty$.

In addition, if μ_k^0 has multiplicity s ($\mu_k^0 = \mu_{k+1}^0 = \cdots = \mu_{k+s-1}^0$), then for any w eigenfunction associated with μ_k^0, with $\|w\|_{H_0} = 1$, there exists a linear combination w^R of eigenfunctions of A_R, $\{w_j^R\}_{j=k}^{j=k+s-1}$ associated with $\{\mu_j^R\}_{j=k}^{j=k+s-1}$ such that

$$\|w^R - \mathscr{R}_R w\|_{H_R} \to 0 \quad \text{as } R \to \infty$$

Lemma 2.2
Let $A : H \longrightarrow H$ be a linear, self-adjoint, positive and compact operator on a Hilbert space H. Let $u \in H$, with $\|u\|_H = 1$ and λ, $r > 0$ such that $\|Au - \lambda u\|_H < r$. Then, there exists an eigenvalue λ_i of A satisfying $|\lambda - \lambda_i| < r$.

2.1. Comparison of the spectra as $R \to \infty$.

The following theorem provides bounds and certain convergence results for the eigenvalues of (6) when $R \to \infty$.

Theorem 2.1
For each fixed i, $i = 1, 2, 3, \ldots$, the sequence of eigenvalues of (7) $\{\lambda_i^R\}_{R>1}$ is an increasing sequence, bounded by λ_i^0 the ith eigenvalue of (12) and, consequently, there is a limit point λ_i^*, λ_i^* satisfying $\lambda_i^* \leqslant \lambda_i^0$. In addition, in the case where $n = 2$, these limit points $\{\lambda_i^*\}_{i=1}^{\infty}$ are eigenvalues of (12).

Proof
Since $\Omega_{R'} \subset \Omega_R$ for $R' < R$, and $\mathscr{V} \subset H^1_{\text{loc}}(\mathbb{R}^n)$, we can assert that $\mathscr{V} \subset \mathscr{V}_R \subset \mathscr{V}_{R'}$. Then, for the ith eigenvalue of (7), the minimax principle allows us to write

$$\lambda_i^{R'} + 1 = \min_{\mathscr{V}_{R',i} \subset \mathscr{V}_{R'}} \max_{u \in \mathscr{V}_{R',i}, u \neq 0} \frac{\int_{\Omega_{R'}} |\nabla u|^2 \, dx + \int_B u^2 \, dx}{\int_B u^2 \, dx}$$

where the minimum is taken over the set of subspaces $\mathscr{V}_{R',i}$ of $\mathscr{V}_{R'}$ with dimension $\dim(\mathscr{V}_{R',i}) = i$. Then, by considering the particular space $\hat{\mathscr{V}}_{R,i} = [u_1^R, u_2^R, \ldots, u_i^R]$, $\{u_j^R\}_{j=1}^{i}$ being the eigenfunctions associated with the ith first eigenvalues of (7), we have that the restriction of these functions to $\Omega_{R'}$ form a subspace of $\mathscr{V}_{R'}$ of dimension i, and, therefore,

$$\lambda_i^{R'} + 1 \leqslant \max_{u \in \hat{\mathscr{V}}_{R,i}, u \neq 0} \frac{\int_{\Omega_{R'}} |\nabla u|^2 \,\mathrm{d}x + \int_B u^2 \,\mathrm{d}x}{\int_B u^2 \,\mathrm{d}x}$$

$$\leqslant \max_{u \in \hat{\mathscr{V}}_{R,i}, u \neq 0} \frac{\int_{\Omega_R} |\nabla u|^2 \,\mathrm{d}x + \int_B u^2 \,\mathrm{d}x}{\int_B u^2 \,\mathrm{d}x} = \lambda_i^R + 1$$

In the same way we prove $\lambda_i^R \leqslant \lambda_i^0$, $\forall R > 1$. Consequently, the sequence $\{\lambda_i^R\}_{R>1}$ increases with R and it is bounded by λ_i^0. Therefore, for fixed i, we have the convergence $\lambda_i^R \to \lambda_i^*$ as $R \to \infty$ for a certain λ_i^* with $\lambda_i^R \leqslant \lambda_i^* \leqslant \lambda_i^0$, and the first assertion in the statement of the theorem holds.

In what follows, we prove that, in the case where $n = 2$, λ_i^* coincides with an eigenvalue of (12) and, specifically $\lambda_1^* = \lambda_1^0$, while this can be false in the case where $n = 3$.

Indeed, let us fix j and denote by $\lambda^* = \lim_{R\to\infty} \lambda_j^R$ and by $\lambda^R = \lambda_j^R$. Let us consider (7) for $\lambda = \lambda^R$ and $u = U_R$: λ^R and $U_R \in \mathscr{V}_R$, $U_R \neq 0$, U_R of norm 1 in $L^2(B)$ and satisfying

$$(\lambda^R + 1) \int_B U_R V \,\mathrm{d}x = \int_{\Omega_R} \nabla U_R \cdot \nabla V \,\mathrm{d}x + \int_B U_R V \,\mathrm{d}x, \quad \forall V \in \mathscr{V}_R \tag{15}$$

Let us consider this sequence of eigenfunctions of (15) $\{U_R\}_R$ associated with $\{\lambda^R\}_R$. From (15) and the boundedness of the sequence $\{\lambda^R\}_R$ we have

$$\|\nabla U_R\|_{(L^2(\Omega_R))^n} + \|U_R\|_{L^2(B)} \leqslant C \tag{16}$$

where C is a constant independent of R. Therefore, when $n = 2$, we can extract a subsequence, still denoted by R, such that for $i = 1, 2$ ($i = 1, 2, 3$ when $n = 3$, respectively), $(\partial U_R / \partial y_i)\mathscr{X}_{\Omega_R} \to f_i$ weakly in $L^2(\mathbb{R}^n)$, as $R \to \infty$, where $\mathscr{X}_{\Omega_R}$ is the characteristic function of Ω_R, and f_i is a certain function $f_i \in L^2(\mathbb{R}^n)$.

Taking into account that for any fixed $K > 1$ and for sufficiently large R, $B(0, K) \subset \Omega_R$, $\mathscr{V}_R \subset H^1(B(0, K))$ and

$$\|U_R\|_{H^1(B(0,K))} \leqslant C(K)$$

where $C(K)$ is a constant depending on K, we can identify the vector function (f_1, f_2) ((f_1, f_2, f_3) when $n = 3$, respectively) with the gradient of a distribution U^*:

$$U^* \in H^1_{\mathrm{loc}}(\mathbb{R}^n), \quad \|U^*\|_{L^2(B)} = 1, \quad U^* \in \mathscr{V}_1 \quad \text{and} \quad \nabla U^* \in (L^2(\mathbb{R}^n))^n \tag{17}$$

Thus, considering (15) for any fixed $V \equiv \varphi \in \mathscr{D}(\mathbb{R}^n)$ and R sufficiently large, and then taking limits as $R \to \infty$, we obtain

$$(\lambda^* + 1) \int_B U^* \varphi \,\mathrm{d}x = \int_{\mathbb{R}^n} \nabla U^* \cdot \nabla \varphi \,\mathrm{d}x + \int_B U^* \varphi \,\mathrm{d}x \tag{18}$$

To conclude the proof of the theorem what remains is to prove that $U^* \in \mathscr{V}$.

Since each eigenfunction U_i^0 of (12) belongs to $\mathscr{V}$, U_i^0 being associated with an eigenvalue λ_i^0, it can be obtained as the limit in $\mathscr{V}$ of smooth functions. Therefore, we can pass to the limit in (18), as φ converge towards U_i^0 in $\mathscr{V}$, to obtain:

$$(\lambda^* + 1)\int_B U^* U_i^0 \,\mathrm{d}x = \int_{\mathbb{R}^n} \nabla U^* \cdot \nabla U_i^0 \,\mathrm{d}x + \int_B U^* U_i^0 \,\mathrm{d}x \tag{19}$$

On the other hand, for the eigenelements (λ_i^0, U_i^0) of problem (11) we can write

$$(\lambda_i^0 + 1)\int_B U_i^0 V \,\mathrm{d}x = \int_B \nabla U_i^0 \cdot \nabla V \,\mathrm{d}x + \int_B U_i^0 V \,\mathrm{d}x + <\mathscr{T} U_i^0, \quad V>_\Gamma, \quad \forall V \in H^1(B) \tag{20}$$

where $\mathscr{T}$ is the Dirichlet–Neumann operator, a linear bounded operator from $H^{1/2}(\Gamma)$ in $H^{-1/2}(\Gamma)$ (see References [1,2] and Section IV.8 in Reference [5]). In particular, by writing (20) for $V = U^* \in H^1(B)$, we have

$$\langle \mathscr{T} U_i^0, U^* \rangle_\Gamma = \int_{\mathbb{R}^{n-} - \bar{B}} \nabla U^{U^*} \cdot \nabla U_i^0 \,\mathrm{d}x \tag{21}$$

where U^{U^*} is the solution of the problem

$$\begin{cases} -\Delta U = 0 \quad \text{in } \mathbb{R}^n - \bar{B} \\ \quad U = U^* \quad \text{on } \Gamma \\ U(x) \to c, \text{ as } |x| \to \infty \quad \text{when } n = 2 \\ U(x) \to 0, \text{ as } |x| \to \infty \quad \text{when } n = 3 \end{cases} \tag{22}$$

Now, because of (18), the function $\tilde{U}^*$ defined as $\tilde{U}^* = U^* - U^{U^*}$, is a harmonic function in $\mathbb{R}^n - \bar{B}$, vanishing on Γ and with a gradient bounded in $L^2(\mathbb{R}^n - \bar{B})$. Then, the behaviour at infinity of this function is given by $\tilde{U}^* = c + O(\frac{1}{|y|})$ when $n = 2, 3$ (see Sections 2.II and 2.III in Reference [9]).

For the dimension $n = 2$ of the space, we apply the Kelvin transform,

$$x' = \frac{x}{|x|^2}, \quad x = \frac{x'}{|x'|^2} \tag{23}$$

to obtain a function $\tilde{U}'^*$, which is a harmonic function inside the domain B' and vanishes on the boundary Γ', where Γ' is the transformed curve of Γ by (23), B' is the region enclosed by Γ' and the function $\tilde{U}'^*$ is defined by

$$\tilde{U}'^*(x') = \tilde{U}^*(x)$$

(see, for instance, Section IV.8 in Reference [5] and Section II.H in Reference [16]). By the uniqueness of the solution of the homogeneous Dirichlet problem in B', $\tilde{U}'^* = 0$ in B', and therefore $\tilde{U}^* = 0$ and $U^* \equiv U^{U^*}$ in $\mathbb{R}^2 - \bar{B}$.

Then, on account of (20) for $V = U^*$, and (21), U^* satisfies

$$(\lambda_i^0 + 1)\int_B U^* U_i^0 \,\mathrm{d}x = \int_{\mathbb{R}^2} \nabla U^* \cdot \nabla U_i^0 \,\mathrm{d}x + \int_B U^* U_i^0 \,\mathrm{d}x \tag{24}$$

and, considering the difference of (24) and (19) we have

$$(\lambda_i^0 - \lambda^*) \int_B U^* U_i^0 \, dx = 0 \tag{25}$$

That is, if we assume that λ^* is not an eigenvalue of (12), then, the function U^* is orthogonal in $L^2(B)$ to the set of eigenfunctions $\{U_i^0\}_{i=1}^{\infty}$ of (12). But, since these eigenfunctions $\{U_i^0\}_{i=1}^{\infty}$ are a basis of $L^2(B)$, U^* is zero in B, which is in contradiction with the fact that it has norm 1 in $L^2(B)$. Therefore, λ^* coincides with an eigenvalue of the set $\{\lambda_i^0\}_{i=1}^{\infty}$, and U^* is an associated eigenfunction $U^* \in \mathscr{V}$, with norm 1 in $L^2(B)$. In this way, the statements of the theorem hold. □

Remark 2.1

Let us note that the limit points λ_i^* in the statement of Theorem 2.1 satisfy (18) with a certain $U^* = U_i^*$ satisfying (17). On the other hand, for the proof of the last part of Theorem 2.1 (see from (23) to (25)) it is essential that $n = 2$. Indeed, for the dimension $n = 3$ of the space, the harmonic functions in $\mathbb{R}^3 - \bar{B}$, which belong to $H^1_{\text{loc}}(\mathbb{R}^3)$ and have a bounded gradient in $L^2(\mathbb{R}^3)$ can converge towards some non-null constant C (cf. Sections II.2 and II.3 in Reference [9]) and Section I.1 in Reference [15]). Therefore, the eigenvalue problem satisfied by (λ^*, U^*) could be posed in a wider space that contains $\mathscr{V}$ (see (17), (18)) and, consequently, to have other eigenvalues different from λ_i^0 (cf. Section I.7 in Reference [5] for the comparison theorem).

Remark 2.2

It should be pointed out that in the statement of Theorem 2.1, for the sequence of problems (6) and for the dimension $n = 2$ of the space, it is a trivial fact that $\lambda_1^R \to \lambda_1^0$ as $R \to \infty$; as a matter of fact, $\lambda_1^0 = \lambda_1^R = 0$ (see Theorem 5.1 to compare). Nevertheless, Theorem 2.1 does not ensure that for $n = 2$ all the eigenvalues of (12) are limit points of eigenvalues of (6); this is proved in Theorem 3.1.

3. SPECTRAL CONVERGENCE FOR $n = 2$

In this section we consider the case where the dimension of the space is $n = 2$ and prove that the asymptotic behaviour of the eigenelements of (6) as $R \to \infty$ is described by the eigenelements of (11) as stated in the following theorem.

Theorem 3.1

Let n be $n = 2$. Let λ_i^R be the eigenvalues of (7) and u_i^R the corresponding eigenfunctions, u_i^R with norm 1 in $\mathscr{V}_R$. For each fixed i, the sequence $\{\lambda_i^R\}_{R>1}$ converges towards the ith eigenvalue λ_i^0 of (12), as $R \to \infty$, and there is conservation of the multiplicity. In addition, for each sequence it is possible to extract a subsequence, still denoted by R, such that the corresponding eigenfunctions, u_i^R, converge towards U_i in $L^2(B)$ (also in $H^1(B)$-weak), where U_i is an eigenfunction associated with the ith eigenvalue of (12), and $\{U_i\}_{i=1}^{\infty}$ form an orthogonal basis of $L^2(B)$.

Proof

The proof of the theorem holds once that we show that $(\lambda_i^R + 1)^{-1}$ $((\lambda_i^0 + 1)^{-1}$, respectively) are the eigenvalues of an operator A_R (A_0, respectively) acting on a Hilbert space H_R

(H_0, respectively), $A_R \in \mathscr{L}(H_R)$ ($A_0 \in \mathscr{L}(H_0)$, respectively), satisfying the following properties C1–C4 in Lemma 2.1.

Indeed, we consider $H_R = H_0 = L^2(B)$, $V = H^1(B)$, A_R and A_0 the linear operators on $L^2(B)$ defined as follows: For $f \in L^2(B)$, $u_f^R = A_R f$ if $u_f^R \in H^1(\Omega_R)$ is the unique solution of the problem

$$\begin{cases} -\Delta u + u = f \quad \text{in } B, \\ \quad -\Delta u = 0 \quad \text{in } \Omega_R - \bar{B} \\ \quad [u] = \left[\dfrac{\partial u}{\partial n}\right] = 0 \quad \text{on } \Gamma \\ \dfrac{\partial u}{\partial n_R} = 0 \quad \text{on } \Gamma_R \end{cases} \tag{26}$$

In the same way, $u_f = A_0 f$ if $u_f \in \mathscr{V}$ is the unique solution of the problem

$$\begin{cases} -\Delta U + U = f \quad \text{in } B \\ \quad -\Delta U = 0 \quad \text{in } \mathbb{R}^2 - \bar{B} \\ \quad [U] = \left[\dfrac{\partial U}{\partial n}\right] = 0 \quad \text{on } \Gamma \\ \quad U(x) \to c, \quad \text{as } |x| \to \infty \end{cases} \tag{27}$$

Obviously, the eigenvalues of A_R are $\{(\lambda_i^R + 1)^{-1}\}_{i=1}^{\infty}$, where $\{\lambda_i^R\}_{i=1}^{\infty}$ are the eigenvalues of (7), and the eigenvalues of A_0 are $\{(\lambda_i^0 + 1)^{-1}\}_{i=1}^{\infty}$, where $\{\lambda_i^0\}_{i=1}^{\infty}$ are the eigenvalues of (12).

Now, considering $\mathscr{R}_R$ the identity operator, $\gamma_0 = 1$, and taking into account that the norms $\|\nabla u\|_{(L^2(\Omega_R))^2} + \|u\|_{L^2(B)}$ and $\|\nabla u\|_{(L^2(\Omega_R))^2} + \|u\|_{L^2(\Omega_R)}$ are equivalent in $H^1(\Omega_R)$, properties C1 and C2 hold.

As regards property C4, for $f_R \in L^2(B)$, with $\|f_R\|_{L^2(B)} \leqslant C$, for C a certain constant independent of R, we consider, the sequence $u_{f_R}^R$ defined by $u_{f_R}^R = A_R f_R$. As known, $u_{f_R}^R$ satisfies the equation

$$\int_{\Omega_R} \nabla u_{f_R}^R \cdot \nabla v \,\mathrm{d}x + \int_B u_{f_R}^R v \,\mathrm{d}x = \int_B f_R v \,\mathrm{d}x, \quad \forall v \in H^1(\Omega_R) \tag{28}$$

Taking $v = u_{f_R}^R$ in (28), we have the bound

$$\int_{\Omega_R} |\nabla u_{f_R}^R|^2 \,\mathrm{d}x + \int_B (u_{f_R}^R)^2 \,\mathrm{d}x \leqslant C$$

where C is a constant independent of R. Then we proceed as in Theorem 2.1 to assert that there is a subsequence, still denoted by R, such that $u_{f_R}^R$ converge in $H^1(B)$-weak, as $R \to \infty$, towards the function u_{f_0} defined by $u_{f_0} = A_0 f_0$, where $f_0 \in L^2(B)$ is the weak limit in $L^2(B)$ of f_R.

Indeed, arguing as in (16)–(18) we obtain a subsequence, still denoted by R, and functions $f_0 \in L^2(B)$ and $u^* \in H^1_{\text{loc}}(\mathbb{R}^2)$ with $\nabla u^* \in (L^2(\mathbb{R}^2))^2$, such that $f_R \to f_0$ in $L^2(B)$-weak, $u_{f_R}^R \to u^*$

in $H^1(B)$-weak, $\mathscr{X}_{\Omega_R}\nabla u^R_{f_R} \to \nabla u^*$ in $(L^2(\mathbb{R}^2))^2$-weak as $R\to\infty$. Hence, considering (28) for any fixed $V\equiv\varphi\in\mathscr{D}(\mathbb{R}^n)$, and for R sufficiently large, and then taking limits as $R\to\infty$, we obtain

$$\int_{\mathbb{R}^n} \nabla u^*\cdot\nabla\varphi\,\mathrm{d}x + \int_B u^*\varphi\,\mathrm{d}x = \int_B f_0\varphi\,\mathrm{d}x \tag{29}$$

Therefore, rewriting the reasoning in (19)–(24), with minor modifications, we obtain that u^* is the unique solution of (22), where we set $U^*\equiv u^*$ and $n=2$. In addition, we can write $u^*\in\mathscr{V}$ and

$$\int_B \nabla u^*\cdot\nabla v\,\mathrm{d}x + \int_B u^* v\,\mathrm{d}x + \langle \mathscr{T}u^*, v\rangle_\Gamma = \int_B f_0 v\,\mathrm{d}x,\quad \forall v\in H^1(B) \tag{30}$$

which is the equation satisfied by the function $u_{f_0}=A_0 f_0$. Thus, by the uniqueness of solution of (30), $u^*=u_{f_0}$ and the weak convergence of the subsequence towards u_{f_0} in $H^1(B)$-weak holds; hence property C4 is proved.

The above reasoning also applies to obtain property C3: it suffices to take $f_R=f$, with $f\in H^1(B)$. Because of the uniqueness of the limits ($f_0\equiv f$, and u_0 the solution of (30)) we obtain the convergence of the whole sequence $u^R_{f_R}$ and the convergence in property C3

$$\|A_R f - A_0 f\|_{L^2(B)}\to 0\quad \text{as } R\to\infty$$

is also true.

In this way, the convergence of the eigenvalues of A_R towards those of A_0 with conservation of the multiplicity holds as well as the convergence of the eigenfunctions stated in Lemma 2.1. Namely, for each eigenfunction u_0 associated with the eigenvalue λ^0 of A_0, λ^0 of multiplicity m_0 and $\|u_0\|_{L^2(B)}=1$, there is $\tilde{u}_R$, $\tilde{u}_R$ being a linear combination of the m_0 eigenfunctions u_R associated with the m_0 eigenvalues λ^R converging towards λ^0, as $R\to\infty$, such that

$$\|\tilde{u}_R - u_0\|_{L^2(B)}\to 0\quad \text{as } R\to\infty$$

As regards the result for the convergence of the subsequence of eigenfunctions u_i^R in the statement of the theorem, it is obtained by taking limits in (7) for $u=u_i^R$, $\lambda=\lambda_i^R$ and fixed $v=\varphi\in\mathscr{D}(\mathbb{R}^2)$: First, using an argument of diagonalization we extract the converging subsequence, still denoted by R, u_i^R converging towards some function U_i in $H^1(B)$-weak, which we identify with an eigenfunction associated with λ_i^0 as in Theorem 2.1 (see (15)–(25)). Then, on account of the orthogonality condition $\int_B u_i^R u_j^R\,\mathrm{d}x=0$ if $i\neq j$, and the normalization for u_i^R (see (9)), $\|u_i^R\|^2_{L^2(B)}=(\lambda_i^R+1)^{-1}$, we prove that the eigenfunctions U_i are orthogonal in $L^2(B)$ and of norm $\|U_i\|^2_{L^2(B)}=(\lambda_i^0+1)^{-1}$. Finally, the fact that they are a basis of $L^2(B)$ holds by contradiction, assuming that there is an eigenfunction U^* associated with a certain λ_j^0, $A_0U^*=(1+\lambda_j^0)^{-1}U^*$ which satisfies

$$\int_B U^*U_i=0,\quad \forall i=1,2,\ldots$$

and proceeding as in Section III.1 of Reference [4] (cf. also Section II.9 of Reference [17]) to obtain that $U^*=0$, which contradicts our assumption. Therefore, the theorem holds. □

4. SPECTRAL CONVERGENCE FOR $n=3$

We consider the asymptotic behaviour of the eigenelements of (6), as $R\to\infty$, for the dimension of the space $n=3$. We assume that, for $R>1$, $\Omega_R=R\Omega_1$ and show that the eigenvalues of (6) converge, as $R\to\infty$, towards those of (13), with the only exception of the first one ($\lambda_1^R=0$), which obviously converge towards 0. Since the technique in Theorems 2.1 and 3.1 cannot be applied (see Remarks 2.1 and 2.2), we are forced to develop an alternative approach which involves the spaces of functions orthogonal to constants in both spaces $L^2(B)$ and $L^2(\Omega_R)$. In particular, the eigenfunctions of (6) do not converge towards true eigenfunctions of (13) and, consequently, they have to be suitably modified by means of an extension operator that we construct (cf. (32)–(34)). In fact, the eigenfunctions of (6) converge in $H^1_{\rm loc}(\mathbb{R}^3)$-weak, as $R\to\infty$, towards functions satisfying (17), (18) (see Remark 4.1). The main convergence results of this section are summarized in Theorem 4.2.

Let us consider the eigenvalue problem (6) in the space $\mathscr{W}_R$ (cf. (8)) whose eigenelements $\{(\Lambda_i^R,U_i^R)\}_{i=1}^{\infty}$ coincide with those of (6) in $H^1(\Omega_R)$ with the only exception of the first one, that is: $\lambda_1^R=0$, $\Lambda_i^R=\lambda_{i+1}^R$, $i=1,2,3,\ldots$, and the corresponding eigenfunction $u_1^R=1$, $U_i^R=u_{i+1}^R$, where (λ_i^R,u_i^R) satisfy (7). We assume that these eigenfunctions are normalized as follows:

$$\|\nabla U_i^R\|^2_{(L^2(\Omega_R))^3}=1 \tag{31}$$

For each eigenfunction U_i^R, let us define the constant C_i^R such that

$$\int_{\Omega_R}(U_i^R-C_i^R)\,\mathrm{d}x=0$$

that is,

$$C_i^R=\frac{1}{|\Omega_R|}\int_{\Omega_R}U_i^R\,\mathrm{d}x \tag{32}$$

Also for each U_i^R, let us denote by $\tilde U_i^R$ the function defined by $\tilde U_i^R=U_i^R-C_i^R$, which obviously satisfies

$$\int_{\Omega_R}\tilde U_i^R\,\mathrm{d}x=0,\quad\text{and}\quad\int_{\Omega_R}|\nabla\tilde U_i^R|^2\,\mathrm{d}x=1$$

Considering $\tilde{\mathscr{W}}_R=\{u\in H^1(\Omega_R)/\int_{\Omega_R}u\,\mathrm{d}x=0\}$, we construct an extension operator $\mathscr{P}_R$ from the space $\tilde{\mathscr{W}}_R$ in $\hat{\mathscr{V}}$, such that

$$\|\mathscr{P}_Ru\|_{\hat{\mathscr{V}}}=\|\nabla\mathscr{P}_Ru\|_{(L^2(\mathbb{R}^3))^3}\leqslant C\|\nabla u\|_{(L^2(\Omega_R))^3},\quad\forall u\in\tilde{\mathscr{W}}_R \tag{33}$$

where C is a constant independent of R and u, and $\hat{\mathscr{V}}$ is the space completion of $\mathscr{D}(\mathbb{R}^n)$ for the Dirichlet norm (see (13)).

In order to define $\mathscr{P}_R$, we consider the extension operator P from $H^1(\Omega_1)$ into $H^1_0(\Omega_2)$, defined by $Pu=u_2$ in $\Omega_2-\bar\Omega_1$, where $u_2\in H^1(\Omega_2-\bar\Omega_1)$ is a lifting of $u|_{\Gamma_1}$, vanishing on Γ_2. We have

$$\|u_2\|_{H^1(\Omega_2-\bar\Omega_1)}\leqslant C\|u\|_{H^{1/2}(\Gamma_1)}\leqslant C\|u\|_{H^1(\Omega_1)}$$

Hence, considering $u \in \tilde{\mathscr{W}}_1$, on account of the Poincaré inequality on this space, we obtain

$$\|\nabla(Pu)\|^2_{(L^2(\Omega_2))^3} \leqslant C\|Pu\|^2_{H^1(\Omega_2)} \leqslant C\|u\|^2_{H^1(\Omega_1)} \leqslant C\|\nabla u\|^2_{(L^2(\Omega_1))^3}, \quad \forall u \in \tilde{\mathscr{W}}_1$$

and

$$\|Pu\|^2_{L^2(\Omega_2)} \leqslant C\|\nabla u\|^2_{(L^2(\Omega_1))^3}, \quad \forall u \in \tilde{\mathscr{W}}_1$$

where C denotes a constant independent of u.

Since we have assumed that $\Omega_R = R\Omega_1$ for $R > 1$, the change of variable $z = x/R$ transforms Ω_R into Ω_1 and the function $w_R(z) = u(x)$ belongs to $\tilde{\mathscr{W}}_1$, provided $u \in \tilde{\mathscr{W}}_R$. Therefore, for $u \in \tilde{\mathscr{W}}_R$ we define $\mathscr{P}_R u(x) = (Pw_R)(x)$ which satisfies $\mathscr{P}_R u \in H^1_0(\Omega_{2R})$, and, from the above inequalities,

$$\|\nabla_x(\mathscr{P}_R u)\|^2_{(L^2(\Omega_{2R}))^3} = R\|\nabla_z(Pw_R)\|^2_{(L^2(\Omega_2))^3} \leqslant CR\|\nabla_z w_R\|^2_{(L^2(\Omega_1))^3} \leqslant C\|\nabla_x u\|^2_{(L^2(\Omega_R))^3}$$

where C is a constant independent of R and u. Then, taking into account that the elements of $H^1_0(\Omega_{2R})$ extended by zero outside Ω_{2R} are elements of $\hat{\mathscr{V}}$, we have constructed the operator $\mathscr{P}_R$ satisfying (33).

Now, considering the functions $\tilde{U}_i^R \in \tilde{\mathscr{W}}_R$, (33) and the normalization (31), for any fixed i, we have that the sequence $\{\mathscr{P}_R \tilde{U}_i^R\}_{R>1}$ is uniformly bounded in $\hat{\mathscr{V}}$ by a constant independent of R. Therefore, taking into account the compact imbedding of $\hat{\mathscr{V}}$ into $L^2(B)$, we can extract a converging sequence, still denoted by R, satisfying

$$\mathscr{P}_R \tilde{U}_i^R \to U_i^* \quad \text{in } \hat{\mathscr{V}} - \text{weak} \quad \text{and} \quad \tilde{U}_i^R \to U_i^* \text{ in } L^2(B), \quad \text{as } R \to \infty \tag{34}$$

The following proposition allows us to prove that we also have the convergence of the corresponding eigenfunctions U_i^R in $H^1_{\text{loc}}(\mathbb{R}^3)$-weak towards $U_i^* + C_i^*$, for C_i^* a constant defined by (35).

Proposition 4.1

For fixed i, let $\{U_i^R\}_{R>1}$ be the eigenfunctions of (6), $U_i^R \in \mathscr{W}_R$, with the norm (31). Let $\{C_i^R\}_{R>1}$ be a sequence of constants defined by (32), and $\tilde{U}_i^R = U_i^R - C_i^R$ such that $\mathscr{P}_R \tilde{U}_i^R \to U_i^*$ in $\hat{\mathscr{V}}$-weak, as $R \to \infty$. Then, $C_i^R \to C_i^*$, as $R \to \infty$, where C_i^* is the constant defined by

$$C_i^* = -\frac{1}{|B|}\int_B U_i^* \, \mathrm{d}x \tag{35}$$

Proof

On account of $\int_B U_i^R \, \mathrm{d}x = 0$, and the definition of C_i^R, we can write

$$C_i^R = \frac{1}{|\Omega_R|}\int_{\Omega_R - \bar{B}} U_i^R \, \mathrm{d}x = \frac{1}{|\Omega_R|}\int_{\Omega_R - \bar{B}} (U_i^R - C_i^R) \, \mathrm{d}x + \frac{1}{|\Omega_R|}\int_{\Omega_R - \bar{B}} C_i^R \, \mathrm{d}x$$

$$= \frac{1}{|\Omega_R|}\int_{\Omega_R} \tilde{U}_i^R \, \mathrm{d}x - \frac{1}{|\Omega_R|}\int_B \tilde{U}_i^R \, \mathrm{d}x + \frac{1}{|\Omega_R|}\int_{\Omega_R - \bar{B}} C_i^R \, \mathrm{d}x$$

Therefore,

$$C_i^R \left(1 - \frac{|\Omega_R - \bar{B}|}{|\Omega_R|}\right) = -\frac{1}{|\Omega_R|}\int_B \tilde{U}_i^R \, \mathrm{d}x$$

and, multiplying this equality by $|\Omega_R|$ and taking limits as $R\to\infty$, on account of the continuous imbedding of $\hat{\mathscr{V}}$ in $L^2(B)$, the right-hand side converge towards $\int_B U_i^*\,\mathrm{d}y$, and the proposition is proved. □

Theorem 4.1
Each eigenvalue of (14) is an accumulation point of eigenvalues of (7).

Proof
Let Λ^* be an eigenvalue of (14) and U^* an eigenfunction associated with Λ^*, $U^*\in\hat{\mathscr{V}}$, U^* of norm 1 in $L^2(B)$. Let C^* be defined by (36) with $U_i^*\equiv U^*$, namely:

$$C^* = -\frac{1}{|B|}\int_B U^*\,\mathrm{d}x \tag{36}$$

Let $\mathscr{H}$ be the Hilbert space $\mathscr{H}=\{u\in L^2(B)/\int_B u\,\mathrm{d}x=0\}$, equipped with the scalar product of $L^2(B)$, and let $\mathscr{W}_R$ be as in (8), $\mathscr{W}_R=\{u\in H^1(\Omega_R)/\int_B u\,\mathrm{d}x=0\}$.

Let A_R be an operator associated with (6) defined as follows: For $f\in\mathscr{H}$, $A_Rf=u_R$ if and only if u_R is the unique solution in $\mathscr{W}_R$ of the problem

$$\begin{cases} -\Delta u = f & \text{in } B\\ -\Delta u = 0 & \text{in } \Omega_R-\bar{B}\\ [u]=\left[\dfrac{\partial u}{\partial n}\right]=0 & \text{on } \Gamma\\ \dfrac{\partial u}{\partial n_R}=0 & \text{on } \Gamma_R \end{cases} \tag{37}$$

On account of the Poincaré inequality in $H^1(B)\cap\mathscr{H}$, it is possible to check that A_R is a positive compact self-adjoint operator on $\mathscr{H}$. The eigenvalues of A_R are $\{(\Lambda_i^R)^{-1}\}_{i=1}^{\infty}$, with Λ_i^R the strictly positive eigenvalues of (6). We apply Lemma 2.2 to function $u=(U^*+C^*)\,\|U^*+C^*\|_{L^2(B)}^{-1}$, parameter $\lambda=(\Lambda^*)^{-1}$ and operator A_R acting on the Hilbert space $\mathscr{H}$, and the statement in the theorem holds from this lemma once that we prove the convergence

$$\left\|A_R(U^*+C^*)-\frac{1}{\Lambda^*}(U^*+C^*)\right\|_{L^2(B)}\to 0 \quad \text{as } R\to\infty \tag{38}$$

Indeed, considering $V_R=A_R(\Lambda^*(U^*+C^*))$, $V_R\in\mathscr{W}_R$ is the solution of (37) when f is replaced by $\Lambda^*(U^*+C^*)$. That is, V_R satisfies

$$\int_{\Omega_R}\nabla V_R\cdot\nabla V\,\mathrm{d}x=\Lambda^*\int_B(U^*+C^*)V\,\mathrm{d}x,\quad \forall V\in H^1(\Omega_R) \tag{39}$$

Taking $V=V_R$ in (39), on account of the Poincaré inequality in $H^1(B)\cap\mathscr{H}$, we have $\int_B V_R^2\,\mathrm{d}y\leqslant C\int_B|\nabla V_R|^2\,\mathrm{d}y$, and we prove

$$\|\nabla V_R\|_{(L^2(\Omega_R))^2}\leqslant C \tag{40}$$

where C is a constant independent of R.

Then, we proceed as in (31)–(34) by considering $\tilde{V}_R = V_R - C_R$, $\tilde{V}_R \in \tilde{\mathscr{W}}_R$ and C_R defined by

$$C_R = \frac{1}{|\Omega_R|} \int_{\Omega_R} V_R \, \mathrm{d}x$$

to obtain that, using operator $\mathscr{P}_R$, the sequence $\{\mathscr{P}_R \tilde{V}_R\}_{R>1}$ is bounded in $\hat{\mathscr{V}}$ by a constant independent of R. Therefore, we can extract a subsequence, still denoted by R, such that $\mathscr{P}_R \tilde{V}_R$ converges in the weak topology of $\hat{\mathscr{V}}$, as $R \to \infty$, towards some function $\hat{U}^* \in \hat{\mathscr{V}}$. We identify $\hat{U}^*$ with the solution of the problem

$$\begin{cases} -\Delta U = \Lambda^* \left(U^* - \dfrac{1}{|B|} \displaystyle\int_B U^* \, \mathrm{d}x \right) \quad \text{in } B \\ -\Delta U = 0 \quad \text{in } \mathbb{R}^3 - \bar{B} \\ \quad [U] = \left[\dfrac{\partial U}{\partial n} \right] = 0 \quad \text{on } \Gamma \\ U(x) \to 0, \quad \text{as } |x| \to \infty \end{cases} \tag{41}$$

writing (39) for $V = \varphi$, with φ any fixed element of $\mathscr{D}(\mathbb{R}^3)$. That is, if we consider R sufficiently large such that $\operatorname{supp}(\varphi) \subset \Omega_R$, we can write

$$\int_{\mathbb{R}^3} \nabla V_R \cdot \nabla \varphi \, \mathrm{d}x = \int_{\mathbb{R}^3} \nabla \tilde{V}_R \cdot \nabla \varphi \, \mathrm{d}x = \Lambda^* \int_B (U^* + C^*) \varphi \, \mathrm{d}x, \quad \forall V \in H^1(\Omega_R)$$

Then, we take limits as $R \to \infty$ and we obtain

$$\int_{\mathbb{R}^3} \nabla U^* \cdot \nabla \varphi \, \mathrm{d}x = \Lambda^* \int_B (\hat{U}^* + C^*) \varphi \, \mathrm{d}x$$

By the uniqueness of the solution of (41) in $\hat{\mathscr{V}}$, (36), and the integral identity satisfied by (Λ^*, U^*) (cf. (14)), we have $\hat{U}^* = U^*$ and the whole sequence $\mathscr{P}_R \tilde{V}_R$ converges weakly in $\hat{\mathscr{V}}$ (also in $H^1_{\mathrm{loc}}(\mathbb{R}^3)$-weak) towards U^* as $R \to \infty$.

Thus, since $\int_B V_R \, \mathrm{d}y = 0$, we proceed also as in Proposition 4.1, where we write V_R and $\tilde{V}_R$ instead of U_i^R and $\tilde{U}_i^R$ to obtain $C_R \to C^*$ with C^* given by (36), as $R \to \infty$. Then, $V_R = \tilde{V}_R + C_R \to U^* + C^*$ in $L^2(B)$, as $R \to \infty$, which shows (38).

Lemma 2.2 ensures the existence of a sequence of eigenvalues $(\Lambda^R_{i(R)})^{-1}$ converging towards $(\Lambda^*)^{-1}$ as $R \to \infty$ and the theorem is proved. □

Theorem 4.2

For fixed i, $i = 1, 2, 3, \ldots$, the sequence of strictly positive eigenvalues of (7) $\{\Lambda_i^R\}_{R>1}$ converges, as $R \to \infty$, towards the ith eigenvalue Λ_i^* of (14). In addition, for the corresponding eigenfunctions $\{U_i^R\}_{R>1}$, it is possible to extract a subsequence, still denoted by R, such that $U_i^R \to U_i^* + C_i^*$ in $L^2(B)$ (also in $H^1(B)$-weak), as $R \to \infty$, where U_i^* is an eigenfunction associated with the ith eigenvalue of (14), and,

$$C_i^* = -\frac{1}{|B|} \int_B U_i^* \, \mathrm{d}x$$

and the sequence $\{U_i^*\}_{i=1}^{\infty}$ satisfies

$$\int_B U_i^* U_j^* \,\mathrm{d}x - \frac{1}{|B|}\int_B U_i^* \,\mathrm{d}x \int_B U_j^* \,\mathrm{d}x = \delta_{i,j}(\Lambda_i^*)^{-1}$$

where $\delta_{i,j}$ is the Kronecker symbol.

Proof

Because of Theorem 2.1, the convergence of Λ_i^R towards some $\lambda_i^* \geqslant 0$ as $R \to \infty$ holds. On the other hand, convergence (34) allows us to extract a subsequence still denoted by R, in such a way that for each $i = 1, 2, \ldots$, we have

$$\Lambda_i^R \to \lambda_i^*, \quad \mathscr{P}_R \tilde{U}_i^R \to U_i^* \quad \text{in } \hat{\mathscr{V}} - \text{weak}, \quad \tilde{U}_i^R \to U_i^* \text{ in } L^2(B), \quad \text{as } R \to \infty \tag{42}$$

First, we prove that, assuming that $\lambda_i^* \neq 0$ and $U_i^* \neq 0$, (λ_i^*, U_i^*) is an eigenelement of (14). Indeed, let us consider the equation satisfied by $(\lambda_i^R, \tilde{U}_i^R)$ (cf. (7))

$$\int_{\Omega_R} \nabla \tilde{U}_i^R \cdot \nabla v \,\mathrm{d}x = \Lambda_i^R \int_B (\tilde{U}_i^R + C_i^R) v \,\mathrm{d}x, \quad \forall v \in H^1(\Omega_R) \tag{43}$$

Let us consider $v = \varphi(x)$ with φ any fixed function $\varphi \in \mathscr{D}(\mathbb{R}^3)$; for sufficiently large R, $\mathrm{supp}(\varphi) \subset \Omega_R$, and we can write

$$\int_{\mathbb{R}^3} \nabla(\mathscr{P}_R \tilde{U}_i^R) \cdot \nabla \varphi \,\mathrm{d}x = \lambda_i^R \int_B (\mathscr{P}_R \tilde{U}_i^R + C_i^R) \varphi \,\mathrm{d}x$$

Then, we take limits in this equation, as $R \to \infty$, and on account of Proposition 4.1 and (42), we have

$$\int_{\mathbb{R}^3} \nabla U_i^* \cdot \nabla \varphi \,\mathrm{d}x = \lambda_i^* \left(\int_B U_i^* \varphi \,\mathrm{d}x - \frac{1}{|B|} \int_B U_i^* \,\mathrm{d}x \int_B \varphi \,\mathrm{d}x \right) \tag{44}$$

and the density of $\mathscr{D}(\mathbb{R}^3)$ in $\hat{\mathscr{V}}$ allows us to assert that (λ_i^*, U_i^*) satisfies (14).

The fact that $\lambda_i^* \neq 0$ and $U_i^* \neq 0$ come from the normalization of the eigenfunctions (31). Indeed, taking $v = \tilde{U}_i^R$ in (43), we can write

$$\Lambda_i^R \int_B (\tilde{U}_i^R + C_i^R)\, \tilde{U}_i^R \,\mathrm{d}x = 1$$

and, Proposition 4.1 and (42) lead us to equality

$$\lambda_i^* \left(\int_B (U_i^*)^2 \,\mathrm{d}x - \frac{1}{|B|} \left(\int_B U_i^* \,\mathrm{d}x \right)^2 \right) = 1$$

which shows that $\lambda_i^* \neq 0$ and $U_i^* \neq 0$ and consequently (λ_i^*, U_i^*) is an eigenelement of (14).

In the same way, the orthogonality condition $\int_{\Omega_R} \nabla U_i^R \cdot \nabla U_j^R \,\mathrm{d}x = 0$, for $i \neq j$, gives the orthogonality condition for $\{U_i^*\}_{i=1}^{\infty}$:

$$\int_B U_i^* U_j^* \,\mathrm{d}x - \frac{1}{|B|} \int_B U_i^* \,\mathrm{d}x \int_B U_j^* \,\mathrm{d}x = 0 \quad \text{for } i \neq j \tag{45}$$

and this allows us to prove that, since the multiplicity of the eigenvalues of (14) is finite, $\lambda_i^* \to \infty$ as $i \to \infty$. That is, because of (45) the eigenfunctions U_i^* associated with a certain

Λ_j are linearly independent functions and they can only be a finite number. Hence, we have an increasing sequence $\{\lambda_i^*\}_{i=1}^{\infty}$, $\lambda_i^* \to \infty$ as $i \to \infty$, $\{\lambda_i^*\}_{i=1}^{\infty} \subset \{\Lambda_i^*\}_{i=1}^{\infty}$.

Next, we prove that $\{\lambda_i^*\}_{i=1}^{\infty} \equiv \{\Lambda_i^*\}_{i=1}^{\infty}$.

Let us assume that there is a certain Λ_j^*, $\lambda_p^* < \Lambda_j^* < \lambda_{p+1}^*$, and because of Theorem 4.1 there is a sequence $\lambda_{i(R)}^R \to \Lambda_j^*$, as $R \to \infty$. Obviously $i(R)$ must tend to ∞ as $R \to \infty$ because, otherwise, $i(R)$ is a fixed integer k for a certain subsequence and then, on account of Theorem 2.1, $\Lambda_j^* = \lambda_k^*$, which is in contradiction with our assumption. But, $i(R) \to \infty$ as $R \to \infty$ means that $i(R) > p+1$ for R sufficiently large and then (cf. Theorem 2.1) $\lambda_{p+1}^R < \lambda_{i(R)}^R$. Taking limits as $R \to \infty$, again because of Theorem 2.1, $\lambda_{p+1}^* \leqslant \Lambda_j^*$ which also contradicts our assumption. Therefore, $\{\lambda_i^*\}_{i=1}^{\infty} \equiv \{\Lambda_i^*\}_{i=1}^{\infty}$.

Finally, since $\{\lambda_i^R\}_{R>1}$ is an increasing sequence, the first assertion of the theorem on the convergence of the positive eigenvalues of (6) towards those of (13) holds. This convergence of the spectrum along with the convergence for the eigenfunctions proved above and the orthogonality condition (45) conclude the proof of the theorem. □

Remark 4.1

Let us note that, since $\mathbf{U}_i^* = U_i^* + C_i$ in Theorem 4.2 satisfies (18) with $\lambda^* = \Lambda_i^*$, results in Theorem 4.2 are not in contradiction with those obtained in the proof of Theorem 2.1. According to Remark 2.1, for $n=3$, the technique in the present section allows us to identify the constant in the condition at the infinity satisfied by the functions U^* in (17), (18) when the corresponding λ^* is a limit point of eigenvalues of (6) as $R \to \infty$.

Remark 4.2

We also observe that results and technique in Section 4 cannot be extended to the case where $n=2$: as a matter of fact, for $n=2$, problem (13) needs a condition at the infinity of the type $U(x) \to C$ as $|x| \to \infty$, for a certain constant C. Thus, in this case, $\lambda = 0$ is an eigenvalue and all the eigenfunctions of this problem are orthogonal to the constants in $L^2(B)$, consequently, the equation in B of problem (13) must be changed for that of (11).

Remark 4.3

Note that the assumption $\Omega_R = R\Omega_1$ performed in this section is only used for the construction of the extension operator $\mathscr{P}_R$ satisfying (33).

5. THE CASE OF SUBDOMAINS OF $\mathbb{R}^{n-}$

As pointed out in Section 1, the results and proofs in Theorems 2.1 and 3.1 extend to the case where B and Ω_R are domains of the lower half space $\mathbb{R}^{n-} = \{x \in \mathbb{R}^n / x_n < 0\}$ and mixed boundary conditions are imposed on the part of the boundary in contact with $\{x_n = 0\}$. We state the main convergence results for these domains in Theorems 5.1 and 5.2; we also outline their proofs.

In contrast, we cannot extend results and proofs in Section 4 for the dimension $n=3$ of the space, and in Section 5.1 we give a convergence result for this dimension, $n=3$, which proves that those in Theorems 5.1 and 5.2 are sharp (see also Remark 5.1). It is worth mentioning that the results in this section are very important when describing the asymptotic behaviour of the low frequencies of certain vibrating systems with many concentrated masses near the boundary (cf. References [6,18]); we refer to Section 6 for a direct application of the results in this section.

Let Ω_R (B, respectively) be any bounded domain of $\mathbb{R}^n$, $n=2,3$, with a Lipschitz boundary $\partial\Omega_R$ (∂B, respectively) and $\Omega_R\subset\{x_n<0\}$ ($B\subset\{x_n<0\}$, respectively). Let Σ_R and Γ_R (T and Γ, respectively) be non-empty parts of the boundary, such that $\partial\Omega_R=\Sigma_R\cup\Gamma_R$ ($\partial B=T\cup\Gamma$, respectively), and Σ_R (T, respectively) is assumed to be in contact with $\{x_n=0\}$. In addition, as in Section 2, we assume that $\mathrm{diam}(\Omega_R)\to\infty$ as $R\to\infty$, that Ω_R contains any fixed domain of $\mathbb{R}^{n-}$, for sufficiently large R, and that for any R,R', such that $1\leqslant R'<R$, we have $\bar{B}\subset\Omega_1$, $\bar{\Omega}_{R'}\subset\Omega_R$.

Let us consider the eigenvalue problem

$$\begin{cases} -\Delta u=\lambda u \quad \text{in } B \\ -\Delta u=0 \quad \text{in } \Omega_R-\bar{B} \\ [u]=\left[\dfrac{\partial u}{\partial n}\right]=0 \quad \text{on } \Gamma \\ u=0 \quad \text{on } T \\ \dfrac{\partial u}{\partial x_n}=0 \quad \text{on } \Sigma_R-\bar{T} \\ \dfrac{\partial u}{\partial n_R}=0 \quad \text{on } \Gamma_R \end{cases} \tag{46}$$

(46) has a strictly positive discrete spectrum, which we denoted by $\{\lambda_i^R\}_{i=1}^\infty$, with the classical convention of repeated eigenvalues, λ_i^R converging to ∞ as $i\to\infty$. Let $\{u_i^R\}_{i=1}^\infty$ be the corresponding eigenfunctions which are assumed to be a basis of $\tilde{\mathscr{V}}_R$, which is the space completion of $\{u\in\mathscr{D}(\bar{\Omega}_R)/u=0 \text{ on } T\}$ for the norm $\|\nabla u\|_{(L^2(\Omega_R))^n}$.

We consider the eigenvalue problem, posed in lower half space $\mathbb{R}^{n-}$,

$$\begin{cases} -\Delta U=\lambda U \quad \text{in } B \\ -\Delta U=0 \quad \text{in } \mathbb{R}^{n-}-\bar{B} \\ [U]=\left[\dfrac{\partial U}{\partial n}\right]=0 \quad \text{on } \Gamma \\ U=0 \quad \text{on } T \\ \dfrac{\partial U}{\partial x_n}=0 \quad \text{on } \{x_n=0\}-\bar{T} \\ U(x)\to c, \text{ as } |x|\to\infty \quad \text{when } n=2 \\ U(x)\to 0, \text{ as } |x|\to\infty \quad \text{when } n=3 \end{cases} \tag{47}$$

which can be written as a standard eigenvalue problem in the spaces $\tilde{\mathscr{V}}\subset L^2(B)$, where $\tilde{\mathscr{V}}$ is the space completion of $\{u\in\mathscr{D}(\overline{\mathbb{R}^{n-}})/u=0 \text{ on } T\}$ for the norm $\|\nabla u\|_{(L^2(\mathbb{R}^{n-}))^n}$. It has a strictly positive discrete spectrum. Let $\{\lambda_i^0\}_{i=1}^\infty$ be the sequence of eigenvalues with the convention of repeated eigenvalues and corresponding eigenfunctions $\{U_i\}_{i=1}^\infty$ which are a orthogonal basis in $\tilde{\mathscr{V}}$ and in $L^2(B)$.

We have the following convergence results.

Theorem 5.1
For each fixed i, $i=1,2,3,\dots$, the sequence of eigenvalues of (46) $\{\lambda_i^R\}_{R>1}$ is an increasing sequence, bounded by λ_i^0 the ith eigenvalue of (47) and, consequently, there is a limit point λ_i^*, $\lambda_i^*=\lim_{R\to\infty}\lambda_i^R$, satisfying $\lambda_i^*\leqslant\lambda_i^0$. In addition, in the case where $n=2$ the limit points $\{\lambda_i^*\}_{i=1}^{\infty}$ are eigenvalues of (47) and, at least, the first one satisfies $\lambda_1^*=\lambda_1^0$.

Proof
The proof of the theorem holds by rewriting the proof of Theorem 2.1 with minor modifications: namely, writing the associated variational formulation for problems (46) and (47) in spaces $\tilde{\mathscr{V}}_R$ and $\tilde{\mathscr{V}}$, respectively, we follow the same steps of the proof of Theorem 2.1 where now we use the Poincaré inequality in $\{u\in H^1(B)/u=0 \text{ on } T\}$, and the fact that a harmonic function on $\mathbb{R}^{n-}-\bar{B}$ with a bounded gradient and satisfying a Neumann condition on $\{x_n=0\}-\bar{T}$ can be extended by reflection to a harmonic function in $\mathbb{R}^n-\bar{\mathbf{B}}$ with a bounded gradient ($\mathbf{B}$ being the corresponding extended domain of B), to obtain the results stated in the theorem. □

Theorem 5.2
Let n be $n=2$. Let λ_i^R be the eigenvalues of (46) and u_i^R the corresponding eigenfunctions with norm 1 in $\tilde{\mathscr{V}}_R$. For fixed i, the sequence $\{\lambda_i^R\}_{i=1}^{\infty}$ converges towards the ith eigenvalue λ_i^0 of (47), when $R\to\infty$, and there is conservation of the multiplicity. In addition, for each sequence it is possible to extract a subsequence, still denoted by R, such that the corresponding eigenfunctions, u_i^R, converge towards U_i in $L^2(B)$ (also in $H^1_{\text{loc}}(\mathbb{R}^2)$-weak), where U_i is an eigenfunction associated with the ith eigenvalue of (47), and $\{U_i\}_{i=1}^{\infty}$ form an orthogonal basis of $L^2(B)$.

Proof
The proof of the theorem holds as that of Theorem 3.1 with minor modifications: namely, we verify properties C1–C4 of Lemma 2.1 for certain operators A_R (A_0, respectively) acting on the Hilbert spaces $H_R=L^2(B)$ ($H_0=L^2(B)$, respectively) and having eigenvalues $\{1/\lambda_i^R\}_{i=1}^{\infty}$ ($\{1/\lambda_i^0\}_{i=1}^{\infty}$, respectively), where $\{\lambda_i^R\}_{i=1}^{\infty}$ are the eigenvalues of (46) ($\{\lambda_i^0\}_{i=1}^{\infty}$ are the eigenvalues of (47)).

These operators A_R are defined as follows: for $f\in L^2(B)$ we define $u_f^R=A_Rf$, where u_f^R is the unique solution in $\tilde{\mathscr{V}}_R$ of the problem

$$\begin{cases} -\Delta u=f \quad \text{in } B \\ -\Delta u=0 \quad \text{in } \Omega_R-\bar{B} \\ [u]=\left[\dfrac{\partial u}{\partial n}\right]=0 \quad \text{on } \Gamma \\ u=0 \quad \text{on } T \\ \dfrac{\partial u}{\partial y_n}=0 \quad \text{on } \Sigma_R-\bar{T} \\ \dfrac{\partial u}{\partial n_R}=0 \quad \text{on } \Gamma_R \end{cases} \tag{48}$$

In the same way, for $f \in L^2(B)$ we define $u_f^0 = A_0 f$, where u_f^0 is the unique solution in $\tilde{\mathscr{V}}$ of the problem

$$\begin{cases} -\Delta U = \lambda U \quad \text{in } B \\ -\Delta U = 0 \quad \text{in } \mathbb{R}^{n-} - \bar{B} \\ [U] = \left[\dfrac{\partial U}{\partial n}\right] = 0 \quad \text{on } \Gamma \\ U = 0 \quad \text{on } T \\ \dfrac{\partial U}{\partial x_n} = 0 \quad \text{on } \{x_n = 0\} - \bar{T} \\ U(x) \to c, \text{ as } |x| \to \infty \quad \text{when } n = 2 \\ U(x) \to 0, \text{ as } |x| \to \infty \quad \text{when } n = 3 \end{cases} \tag{49}$$

Then, considering $\mathscr{R}_R$ the identity operator in $L^2(B)$, and rewriting the proof of Theorem 3.1 with minor modifications, we verify properties C1–C4 of Lemma 2.1 and the theorem holds. □

Remark 5.1
It should be noted that the proofs of Theorems 5.1–5.2 imply that the limit eigenelements (λ^*, U^*) of the eigenelements of (46), $(\lambda^*, U^*) \in \mathbb{R} \times H^1(B)$, satisfy

$$\lambda^* \int_B U^* \varphi \, dx = \int_{\mathbb{R}^{n-}} \nabla U^* \cdot \nabla \varphi \, dx \tag{50}$$

for any fixed φ, $\varphi \in \mathscr{D}(\mathbb{R}^{n-})$, $\varphi = 0$ on T, and $U^* \neq 0$

$$U^* \in H^1_{\text{loc}}(\mathbb{R}^{n-}), \quad U^*|_T = 0, \quad U^* \in \tilde{\mathscr{V}}_1 \text{ and } \nabla U^* \in (L^2(\mathbb{R}^{n-}))^n \tag{51}$$

5.1. Remarks on the spectral convergence for $n = 3$

Let us observe that, on account of the results in Theorem 5.1, $\{\lambda_1^R\}_{R>1}$ is an increasing sequence such that $\lambda_1^R \leqslant \lambda_1^0$, and, therefore its limit, as $R \to \infty$, also satisfies $\lambda_1^* \leqslant \lambda_1^0$. On the other hand, using the minimax principle it is not difficult to derive that $\lambda_1^R < \lambda_1^0$ for all $R > 1$. In this section, we prove that $\lambda_1^* < \lambda_1^0$ can also happen. This fact depends, for instance, on the domain B, and it turns out to be very important for the study of vibrating systems with concentrated masses (see Section 6).

For $n = 3$, in the case where λ is not an eigenvalue of problem (47), let us consider the function $F(\lambda)$ defined by

$$F(\lambda) = -\left\langle \left.\frac{\partial V^\lambda}{\partial n}\right|_\Gamma, 1 \right\rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} \tag{52}$$

with $V^\lambda = \hat{U} + W$, where $\hat{U}$ is the solution of the non-homogeneous problem associated with (47) when the equation in B is replaced by

$$-\Delta U = \lambda U + \lambda W \quad \text{in } B$$

and W is the solution of the problem

$$\begin{cases} -\Delta U = 0 & \text{in } \mathbb{R}^{3-} \\ U = 0 & \text{on } T \\ \dfrac{\partial U}{\partial x_3} = 0 & \text{on } \{x_3 = 0\} - \bar{T} \\ U(x) \to 1 & \text{as } |y| \to \infty,\ x_3 < 0 \end{cases} \tag{53}$$

We state some properties of function F in the following lemma (cf. Reference [10] for the proof) which allow to prove Theorem 5.3 below.

Lemma 5.1
The function $F(\lambda)$ defined by (52) is a meromorphic function with infinite positive real poles $\{v_i\}_{i=1}^{\infty}$ which are eigenvalues of (47). Moreover, $F(\lambda)$ is negative for negative λ and, for each $i \in \mathbb{N}$ and real λ, it satisfies

$$\lim_{\lambda \to v_i^+} F(\lambda) = -\infty \quad \text{and} \quad \lim_{\lambda \to v_i^-} F(\lambda) = +\infty$$

In addition, in the case where the solution W of (53) is not orthogonal in $L^2(B)$ to the eigenspace associated with λ_1^0 (λ_1^0 the first eigenvalue of (47)), $v_1 \equiv \lambda_1^0$.

Theorem 5.3
In the case where the solution W of (53) is not orthogonal in $L^2(B)$ to the eigenspace associated with λ_1^0, λ_1^R converge, as $R \to \infty$, towards some λ^* which satisfies $\lambda^* < \lambda_1^0$.

Proof
The fact that λ_1^R converge towards some positive $\lambda^* \leqslant \lambda_1^0$ is a consequence of Theorem 5.1. In order to prove that they are different numbers we consider the function $F(\lambda)$ in (52), which is also defined by

$$F(\lambda) = \lambda \int_B (U^\lambda - 1)^2 \, dx - \int_{\mathbb{R}^{3-}} |\nabla U^\lambda|^2 \, dx \tag{54}$$

(see Reference [10] to prove the equivalence of both definitions), where $U^\lambda = 1 - V^\lambda$ is the solution of

$$\begin{cases} -\Delta U = \lambda U - \lambda & \text{in } B \\ -\Delta U = 0 & \text{in } \mathbb{R}^{3-} - \bar{B} \\ [U] = \left[\dfrac{\partial U}{\partial n}\right] = 0 & \text{on } \Gamma \\ U = 1 & \text{on } T \\ \dfrac{\partial U}{\partial x_3} = 0 & \text{on } \{x_3 = 0\} - \bar{T} \\ U(x) \to 0 & \text{as } |x| \to \infty,\ x_3 < 0 \end{cases} \tag{55}$$

Then, considering $\lambda = \lambda_1^R$ in (54) we can write

$$\frac{F(\lambda_1^R)}{\int_B (V^{\lambda_1^R})^2\,\mathrm{d}x} \leqslant \lambda_1^R - \frac{\int_{\Omega_R} |\nabla V^{\lambda_1^R}|^2\,\mathrm{d}x}{\int_B (V^{\lambda_1^R})^2\,\mathrm{d}x}$$

Now, applying the minimax principle, since $V^{\lambda_1^R} \in \tilde{\mathscr{V}}_R$, we have

$$\lambda_1^R = \frac{\int_{\Omega_R} |\nabla U_1^R|^2\,\mathrm{d}x}{\int_B (U_1^R)^2\,\mathrm{d}x} \leqslant \frac{\int_{\Omega_R} |\nabla V^{\lambda_1^R}|^2\,\mathrm{d}x}{\int_B (V^{\lambda_1^R})^2\,\mathrm{d}x}$$

where U_R^1 denotes an eigenfunction of (46) associated with λ_1^R. Therefore,

$$\frac{F(\lambda_1^R)}{\int_B (V^{\lambda_1^R})^2\,\mathrm{d}x} \leqslant 0$$

and also $F(\lambda_1^R) \leqslant 0$. Because of Theorem 5.3, the continuity of $F(\lambda)$ for $\lambda \neq \lambda_i^0$ implies $F(\lambda_1^R) \to F(\lambda^*) \leqslant 0$ as $\lambda_1^R \to \lambda^* \leqslant \lambda_1^0$. That is, $F(\lambda^*) \leqslant 0$ and, because of the last assertion in Lemma 5.1, $\lim_{\lambda \to \lambda_1^0-} F(\lambda) = +\infty$ and λ^* cannot be the first eigenvalue of (47). □

Remark 5.2
We emphasize that Theorem 5.3 proves that the first eigenvalue λ_1^R converges towards λ_1^0 the first eigenvalue of problem (47), as $R \to \infty$, only for the dimension $n = 2$ of the space.

In contrast, in the case where a Dirichlet condition is considered on Γ_R instead of a Neumann one, for both problems (6) and (46), and for both dimensions $n = 2$ and 3, the first eigenvalue λ_1^R converges towards λ_1^0, as $R \to \infty$. This can be proved using the technique in Section VII.11 of Reference [5], Section III.1 of References [4,10,12], as has already been noted in Section 1 for the limiting problem of (1), (2) as $\mathrm{diam}(\omega) \to \infty$.

6. ON VIBRATING SYSTEMS WITH MANY CONCENTRATED MASSES NEAR THE BOUNDARY

In this section we apply properties in previous sections to derive the convergence of the spectrum for an eigenvalue problem appearing in the literature involved with the study of vibrating systems with many concentrated masses. As it has been noted in Section 1, these vibrating systems have been highly studied by different authors: see Reference [6] for references. They consider the asymptotic behaviour, as $\varepsilon \to 0$, of the vibrations of a body that contains many small regions of diameter $O(\varepsilon)$ where the density is higher than elsewhere. The density takes the value $O(\varepsilon^{-m})$ in these regions and $O(1)$ outside. The small regions are the so-called *concentrated masses* and are placed near the boundary at a distance η between them. ε and η are parameters converging towards zero, and $m > 0$. Different results have been obtained in References [10–14], for different boundary conditions and shapes of the domains Ω and B, depending on the dimension of the space n, the value of m and the relation between ε and η.

Here, we consider alternating mixed boundary conditions (cf. problem (56)), $n = 2, 3$, $\varepsilon \ll \eta$ and $m > 2$. We prove results in Theorem 6.1, which have been announced in Reference [6] without any proof. These results are essential in order to describe the convergence of the re-scaled eigenvalues $\lambda_i^\varepsilon \varepsilon^{2-m}$ of (56) (see Remarks 6.1 and 6.2).

Let Ω be an open bounded domain of $\mathbb{R}^{n-}$, $n=2,3$, with a Lipschitz boundary $\partial\Omega$. Let Σ and Γ_Ω be non-empty parts of the boundary, such that $\partial\Omega = \bar{\Sigma} \cup \bar{\Gamma}_\Omega$, and Σ is assumed to be in contact with $\{x_n = 0\}$. Let ε and η be two small parameters such that $\varepsilon \ll \eta$ and $\eta = \eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let us assume that the parameter m is a real number, $m > 2$.

For $n=2$, let B be an open bounded domain of the auxiliary space $\mathbb{R}^{n-} = \{y \in \mathbb{R}^n / y_n < 0\}$, $\mathbb{R}^n$ with coordinates y_1, y_2 if $n=2$ (y_1, y_2, y_3 if $n=3$). Let ∂B be the boundary of B, ∂B a Lipschitz boundary, $\partial B = \bar{T} \cup \bar{\Gamma}$, where T is the part lying on $\{y_n = 0\}$. Let B^ε (and similarly T^ε, Γ^ε) denote its homothetic εB (εT, $\varepsilon\Gamma$). Let B_k^ε (and similarly T_k^ε, Γ_k^ε) denote the domain obtained by translation of the previous B^ε (T^ε, Γ^ε) centred at the point $\tilde{x}_k$ of Σ at distance η between them. k is a parameter ranging from 1 to $N(\varepsilon)$, $k \in \mathbb{N}$. $N(\varepsilon)$ denotes the number of B_k^ε contained in Ω; $N(\varepsilon)$ is of order $O(1/\eta)$ when $n=2$ and $O(1/\eta^2)$ when $n=3$.

We consider the eigenvalue problem

$$\begin{cases} -\Delta u^\varepsilon = \rho^\varepsilon \lambda^\varepsilon u^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \Gamma_\Omega \cup \bigcup T^\varepsilon \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \Sigma - \overline{\bigcup T^\varepsilon} \end{cases} \tag{56}$$

where $\rho^\varepsilon = \rho^\varepsilon(x)$ is the function defined as

$$\rho^\varepsilon(x) = \begin{cases} \dfrac{1}{\varepsilon^m} & \text{if } x \in \bigcup B^\varepsilon \\ 1 & \text{if } x \in \Omega - \overline{\bigcup B^\varepsilon} \end{cases} \tag{57}$$

The symbol $\bigcup$ is extended, for fixed ε, to all the regions B_k^ε contained in Ω.

As is well known, problem (56) has a positive discrete spectrum. For fixed ε, let $\{\lambda_i^\varepsilon\}_{i=1}^\infty$ be the sequence of eigenvalues of (56), converging to ∞, with the classical convention of repeated eigenvalues. It has been proved (see References [10–12]) that they satisfy the estimates

$$C\varepsilon^{m-2} \leqslant \lambda_i^\varepsilon \leqslant C_i \varepsilon^{m-2} \tag{58}$$

where C is a constant independent of ε and i, and, C_i is a constant independent of ε. Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the corresponding sequence of eigenfunctions which are assumed to be an orthonormal basis of the space $\mathbf{V}^\varepsilon$, where $\mathbf{V}^\varepsilon$ is the completion of $\{u \in \mathscr{D}(\bar{\Omega}) / u = 0 \text{ on } \Gamma_\Omega \cup \bigcup T^\varepsilon\}$ in the topology of $H^1(\Omega)$.

We emphasize that problem (47) is the so-called *local problem* which is involved with the *low frequencies* of (56) (i.e. the eigenvalues in (58)) and the *local vibrations* of the concentrated masses. See References [6,13,18] for the corresponding eigenfunctions. From (56) we reach (47) by performing the change of variable in a neighbourhood of each region B_k^ε

$$y = \frac{x - \tilde{x}_k}{\varepsilon} \tag{59}$$

where, obviously, the variable x must be replaced by y in formulas of (47). We also observe that other geometries for B and Ω could be considered (cf., for instance Reference [6]).

Let us introduce $\tilde{\varphi}^\varepsilon(y)$, a function defined depending on the value of n. For $n=2$, we consider $R_\varepsilon=\sqrt{1+\eta/4\varepsilon}$ and we define

$$\tilde{\varphi}^\varepsilon(y)=\begin{cases}1 & \text{if } |y|\leqslant R_\varepsilon\\ 1-\dfrac{\ln|y|-\ln R_\varepsilon}{\ln R_\varepsilon} & \text{if } R_\varepsilon\leqslant|y|\leqslant R_\varepsilon^2\\ 0 & \text{if } |y|\geqslant R_\varepsilon^2\end{cases} \tag{60}$$

For $n=3$, we consider $\tilde{\varphi}^\varepsilon$ as a smooth function which takes the value 1 in the semi-ball of radius $((\varepsilon+\eta/8)/\varepsilon)$, $B((\varepsilon+\eta/8)/\varepsilon)$, and is zero outside the semi-ball of radius $((\varepsilon+\eta/4)/\varepsilon)$, $B(\varepsilon+\eta/4)/\varepsilon)$:

$$\tilde{\varphi}^\varepsilon(y)=\varphi\left(2\frac{|\varepsilon y|-\varepsilon}{\eta}\right) \tag{61}$$

where $\varphi\in C^\infty[0,1]$, $0\leqslant\varphi\leqslant 1$, $\varphi=1$ in $[0,1/4]$ and $\text{supp}(\varphi)\subset[0,1/2]$.

Considering $\{U_p^0\}_{p=1}^\infty$ the set of eigenfunctions of (47), it has been proved in Reference [12] when $n=2$ (in Reference [10] when $n=3$) that $\tau_x(U_p^0\tilde{\varphi}^\varepsilon)\in\mathbf{V}^\varepsilon$, where τ_x means the change of variable (59) from y to x, and

$$U_p^0\tilde{\varphi}^\varepsilon\xrightarrow{\varepsilon\to 0}U_p^0\quad\text{in }\tilde{\mathscr{V}} \tag{62}$$

In order to prove the following theorem, we observe that in the case where $n=3$ and $\lim_{\varepsilon\to 0}\varepsilon\eta^{-2}=0$, function in (61) can be replaced by another one with a shorter support, namely,

$$\tilde{\varphi}^\varepsilon(y)=\varphi\left(2\frac{|\varepsilon y|-\varepsilon}{\sqrt{\varepsilon}}\right) \tag{63}$$

which also satisfies (62) for the same φ as in (61).

Theorem 6.1

- For $n=2,3$, let λ_1^ε and λ_1^0 be the first eigenvalues of (56) and (47), respectively. Then, there exists a constant $\lambda^*\leqslant\lambda_1^0$ and a sequence o_ε, $o_\varepsilon\to 0$ as $\varepsilon\to 0$, such that $\lambda^*\leqslant\lambda_1^\varepsilon/\varepsilon^{m-2}\leqslant\lambda_1^0+o_\varepsilon$.
- For $n=2$, the sequence $\lambda_1^\varepsilon/\varepsilon^{m-2}$ converges towards the first eigenvalue λ_1^0 of the local problem (47) as $\varepsilon\to 0$.

Proof

The minimax principle gives

$$\frac{\lambda_1^\varepsilon}{\varepsilon^{m-2}}=\min_{u\in\mathbf{V}^\varepsilon,u\neq 0}\frac{\int_\Omega|\nabla u|^2\,\mathrm{d}x}{\varepsilon^{-2}\int_{\bigcup B^\varepsilon}u^2\,\mathrm{d}x+\varepsilon^{m-2}\int_{\Omega-\overline{\bigcup B^\varepsilon}}u^2\,\mathrm{d}x} \tag{64}$$

Then, considering the change of variable (59) from x to y, U_1^0 an eigenfunction of (47) associated with the first eigenvalue λ_1^0, $\tilde{\varphi}^\varepsilon(y)$ defined by (60) when $n=2$ and (61) when $n=3$

(respectively, (63) when $n=3$ and $\lim_{\varepsilon\to 0}\varepsilon\eta^{-2}=0$), the convergence (62), the Poincaré inequality on $\{U\in H^1(B(0,1+\eta/4\varepsilon))/U=0 \text{ on } T\}$ (respectively, on $\{U\in H^1(B(0,1+1/4\sqrt{\varepsilon}))/U=0 \text{ on } T\}$), and the integral equality for the eigenelement (λ_1^0, U_1^0) of (47), we can write

$$\frac{\lambda_1^\varepsilon}{\varepsilon^{m-2}} \leqslant \frac{\int_{\mathbb{R}^{n-}} |\nabla_y(U_1^0\tilde{\varphi}^\varepsilon)|^2\,\mathrm{d}y}{\int_B (U_1^0\tilde{\varphi}^\varepsilon)^2\,\mathrm{d}y + \varepsilon^m \int_{\mathbb{R}^{n-}} (U_1^0\tilde{\varphi}^\varepsilon)^2\,\mathrm{d}y} \leqslant \lambda_1^0 + o_\varepsilon \tag{65}$$

where $o_\varepsilon \to 0$ as $\varepsilon\to 0$.

For $R>1$ we denote by $B(0,R)$ the semi-ball in $\mathbb{R}^{n-}$ of radius R. We use the notation of Section 5, that is, we consider $\Omega_R = B(0,R)$, $\tilde{\mathscr{V}}_R$ denotes the space completion of $\{U\in\mathscr{D}(\overline{B(0,R)})/U=0 \text{ on } T\}$ for the norm $\|\nabla_y U\|_{(L^2(B(0,R)))^n}$, and we apply the Poincaré inequality on $\tilde{\mathscr{V}}_R$, to obtain

$$\int_B U^2\,\mathrm{d}y \leqslant (\lambda_1^R)^{-1}\int_{B(0,R)} |\nabla_y U|^2\,\mathrm{d}y, \quad \forall U\in\tilde{\mathscr{V}}_R \tag{66}$$

where (λ_1^R) denotes the first eigenvalue of problem (46).

Now, considering (64), we perform the change of variable (59) on each B_k^ε, we apply (66) taking into account that $B(0,R\varepsilon)\subset B(\varepsilon+\eta/4)$, for sufficiently small $\varepsilon(\varepsilon\leqslant\varepsilon_R)$, and we have

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{B_k^\varepsilon} u(x)^2\,\mathrm{d}x &= \varepsilon^{n-2}\int_B u(y)^2\,\mathrm{d}y \leqslant \varepsilon^{n-2}(\lambda_1^R)^{-1}\int_{B(0,R)} |\nabla_y u|^2\,\mathrm{d}y\\ &= (\lambda_1^R)^{-1}\int_{B(0,R\varepsilon)} |\nabla_x u|^2\,\mathrm{d}x \leqslant (\lambda_1^R)^{-1}\int_{B(\varepsilon+\eta/4)} |\nabla_x u|^2\,\mathrm{d}x, \quad \forall v\in\mathbf{V}^\varepsilon\end{aligned}$$

Therefore,

$$\frac{1}{\varepsilon^2}\int_{\bigcup B^\varepsilon} u^2\,\mathrm{d}x \leqslant (\lambda_1^R)^{-1}\int_\Omega |\nabla_x u|^2\,\mathrm{d}x, \quad \forall u\in\mathbf{V}^\varepsilon \tag{67}$$

We also apply the Poincaré inequality on $\mathbf{V}^\varepsilon$

$$\int_\Omega u^2\,\mathrm{d}x \leqslant C\int_\Omega |\nabla_x u|^2\,\mathrm{d}x, \quad \forall u\in\mathbf{V}^\varepsilon \tag{68}$$

Then, from (67), (68) and (64), we obtain

$$\frac{\lambda_1^R}{C\varepsilon^{m-2}\lambda_1^R + 1} \leqslant \frac{\lambda_1^\varepsilon}{\varepsilon^{m-2}}, \quad \forall\varepsilon\leqslant\varepsilon_R \tag{69}$$

and, on account of (65), the first statement of the theorem is proved.

In addition, because of (65) and (69), for any fixed $R>1$, we have

$$\lambda_1^R \leqslant \underline{\lim}_{\varepsilon\to 0}\frac{\lambda_1^\varepsilon}{\varepsilon^{m-2}} \leqslant \overline{\lim}_{\varepsilon\to 0}\frac{\lambda_1^\varepsilon}{\varepsilon^{m-2}} \leqslant \lambda_1^0 \tag{70}$$

As regards the second statement, in the case where $n=2$, we prove that the sequence $\lambda_1^\varepsilon/\varepsilon^{m-2}$ converges towards λ_1^0, as $\varepsilon\to 0$, using (70) and the convergence $\lambda_1^R\to\lambda_1^0$ as $R\to\infty$, which holds from the last statement in Theorem 5.1. □

Remark 6.1
The result in the second statement of Theorem 6.1 for $n=2$ can be completed using results on the total multiplicity of the eigenvalues converging towards λ_1^0 (cf. References [6,13,18]); in fact, it can be proved: *For* $n=2$ *and for fixed* i, $i=1,2,3,\ldots$, *the sequence* $\lambda_i^\varepsilon/\varepsilon^{m-2}$ *converges towards the first eigenvalue* λ_1^0 *of the local problem* (52).

Remark 6.2
We observe that the result in the second statement of Theorem 6.1 is sharp. As a matter of fact, it has been proved in Reference [10] that for $n=3$, and for the relations between ε and η such that $\lim_{\varepsilon\to 0}(\varepsilon/\eta^2)>0$, there are other accumulation points of $\lambda_i^\varepsilon/\varepsilon^{m-2}$ that can be smaller than the first eigenvalue λ_1^0 of the local problem (47). The previously mentioned proof in Reference [10] is based on certain properties of the function F defined in (52) (cf. Lemma 5.1), and this result is in good agreement with that in Theorem 5.3.

Remark 6.3
Let us note that in problem (56) a Dirichlet condition is imposed on the boundary Γ_Ω and (47) appears as the local problem. The same can be said if we consider the concentrated masses inside Ω, a Dirichlet condition on $\partial\Omega$, and the resulting local problem (11). Nevertheless, it should be pointed out that in this case the first eigenvalue of (11) is $\lambda_1^0=0$ when $n=2$, and, according to Remark 6.1, a further study should be performed on the problem with many concentrated masses inside Ω.

On the other hand, results in Section 6 show that the limit behaviour of the spectrum of (1), (3) is also important when describing spectral convergence for vibrating systems with concentrated masses, when a Dirichlet condition is prescribed on the boundary of the domain Ω, instead of a Neumann condition.

Remark 6.4
The vibrations of a system with one single (or a fixed number) concentrated mass inside Ω, and a Neumann condition on the whole boundary of Ω, has been approached in Reference [19] for the dimension $n=3$ of the space, using asymptotic expansions. The limit problem for the spectrum is (13) for certain values of m. Results and techniques in Reference [19] are different from those in this paper, and, it seems reasonable to think that the results in this paper can be used in the case where a Neumann condition is imposed on Γ_Ω for (56) or on the whole $\partial\Omega$.

Remark 6.5
It should be noted that the first eigenvalue λ_1^R of (46) converges towards some positive λ_1^*, as $R\to\infty$, and the fact that this value λ_1^* is (or is not, respectively) the first eigenvalue of the local problem (47) seems to be deeply involved with the fact that the low frequencies of (56) give rise only to local vibrations of the concentrated masses (or they also give rise to global vibrations, respectively).

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