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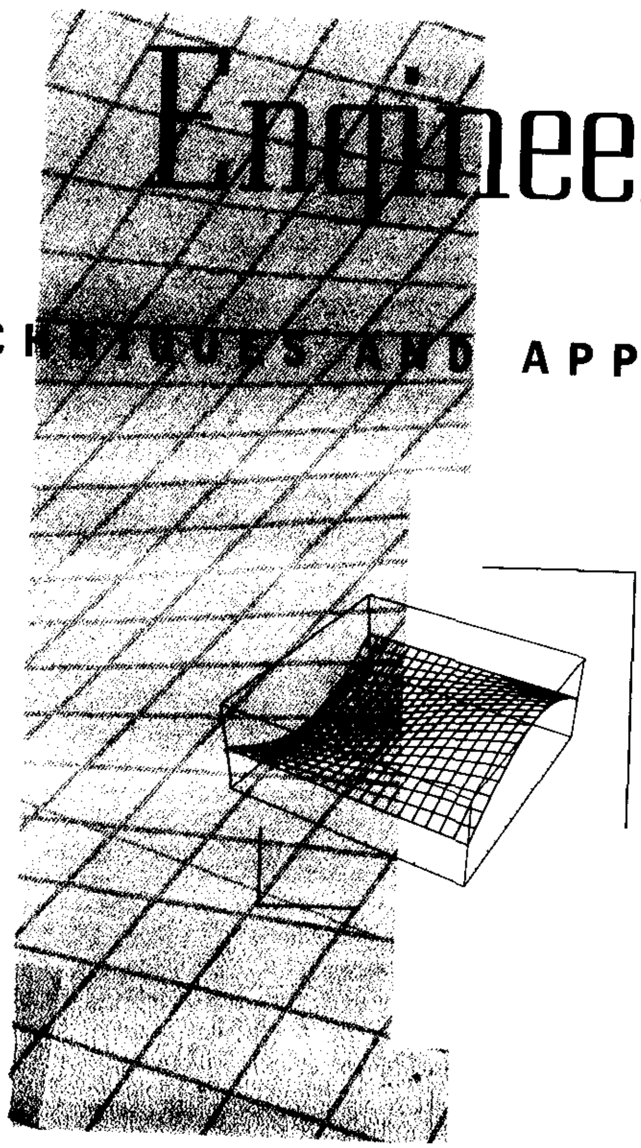

TECHNIQUES AND APPLICATIONS

C. Constanda, S. Potapenko, Editors

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The physical world is studied by means of mathematical models, which consist of differential, integral, and integro-differential equations accompanied by a large assortment of initial and boundary conditions. In certain circumstances, such models yield exact analytic solutions. When they do not, they are solved numerically by means of various approximation schemes. Whether analytic or numerical, these solutions share a common feature: they are constructed by means of the powerful tool of integration—the focus of this self-contained book.

An outgrowth of the Ninth International Conference on Integral Methods in Science and Engineering, this work illustrates the application of integral methods to diverse problems in mathematics, physics, biology, and engineering. The thirty two chapters of the book, written by scientists with established credentials in their fields, contain state-of-the-art information on current research in a variety of important practical disciplines. The problems examined arise in real-life processes and phenomena, and the solution techniques range from theoretical integral equations to finite and boundary elements.

Specific topics covered include spectral computations, atmospheric pollutant dispersion, vibration of drilling masts, bending of thermoelastic plates, homogenization, equilibria in nonlinear elasticity, modeling of syringomyelia, fractional diffusion equations, operators on Lipschitz domains, systems with concentrated masses, transmission problems, equilibrium shape of axisymmetric vesicles, boundary layer theory, and many more.

Integral Methods in Science and Engineering is a useful and practical guide to a variety of topics of interest to pure and applied mathematicians, physicists, biologists, and civil and mechanical engineers, at both the professional and graduate student level.

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Integral Methods in Science and Engineering

Techniques and Applications

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Preface

The international conferences with the generic title of Integral Methods in Science and Engineering (IMSE) are a forum where academics and other researchers who rely significantly on (analytic or numerical) integration methods in their investigations present their newest results and exchange ideas related to future projects.

The first two conferences in this series, IMSE1985 and IMSE1990, were held at the University of Texas at Arlington under the chairmanship of Fred Payne. At the 1990 meeting, the IMSE consortium was created for the purpose of organizing these conferences under the guidance of an International Steering Committee. Subsequently, IMSE1993 took place at Tohoku University, Sendai, Japan, IMSE1996 at the University of Oulu, Finland, IMSE1998 at Michigan Technological University, Houghton, MI, USA, IMSE2000 in Banff, AB, Canada, IMSE2002 at the University of Saint-Étienne, France, and IMSE2004 at the University of Central Florida, Orlando, FL, USA. The IMSE conferences have now become recognized as an important platform for scientists and engineers working with integral methods to contribute directly to the expansion and practical application of a general, elegant, and powerful class of mathematical techniques.

A remarkable feature of all IMSE conferences is their socially enjoyable atmosphere of professionalism and camaraderie. Continuing this trend, IMSE2006, organized at Niagara Falls, ON, Canada, by the Department of Civil and Environmental Engineering and the Department of Applied Mathematics of the University of Waterloo, was yet another successful event in the history of the IMSE consortium, for which the participants wish to express their thanks to the Local Organizing Committee:

Stanislav Potapenko (University of Waterloo), *Chairman;*
Peter Schiavone (University of Alberta);
Graham Gladwell (University of Waterloo);
Les Sudak (University of Calgary);
Siv Sivaloganathan (University of Waterloo).

The organizers and the participants also wish to acknowledge the financial support received from the Faculty of Engineering and the Department of Applied Mathematics, University of Waterloo and the Department of Mechanical Engineering, University of Alberta.

The next IMSE conference will be held in July 2008 in Santander, Spain. Details concerning this event are posted on the conference web page, http://www.imse08.unican.es.

This volume contains 2 invited papers and 30 contributed papers accepted after peer review. The papers are arranged in alphabetical order by (first) author's name.

The editors would like to record their thanks to the referees for their willingness to review the papers, and to the staff at Birkhäuser-Boston, who have handled the publication process with their customary patience and efficiency.

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21

On Quasimodes for Spectral Problems Arising in Vibrating Systems with Concentrated Masses

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21.1 Introduction

*Q*uasimodes for positive, symmetric, and compact operators on Hilbert spaces arise often in the literature in the description of behavior at high-frequency vibrations (see, for example, [Arn72], [BB91], [Laz99], and [Per03]). Roughly speaking, the quasimodes u can be defined as functions approaching a certain linear combination of eigenfunctions associated with eigenvalues in "small intervals" $[\lambda - r, \lambda + r]$. Their usefulness in describing asymptotics for low frequency vibrations in certain singularly-perturbed spectral problems has been made clear recently in many papers (see [LP03], [Per04], [Per05], [Per06], and [Per07]).

In this chapter, we consider a vibrating system with concentrated masses. Namely, we consider the vibrations of a body occupying a domain Ω of $\mathbb{R}^n$, $n = 2, 3$, that contains many small regions (B^ε) of high density—so-called *concentrated masses*—near the boundary. The small parameter ε deals with the size of the masses, their number, and their densities. The asymptotic behavior, as $\varepsilon \to 0$, of the eigenelements $(\lambda^\varepsilon, u^\varepsilon)$ of the corresponding spectral problem, namely problem (21.3), when $\lambda^\varepsilon = O(\varepsilon^{m-2})$, has been treated in [Per04] (see [LP03] for a substantial list of references on the subject). Here, considering the hyperbolic problem (21.15), we provide estimates for the time t in which certain *standing waves* approach time-dependent solutions when the initial data are quasimodes. Also, precise bounds for the discrepancies between the solutions and standing waves in suitable Hilbert spaces are provided. The results can be extended to high-frequency vibrations.

It should be mentioned that in certain problems arising in spectral perturbation theory, the eigenfunctions associated with low frequencies give rise to vibrations of the system that are concentrated asymptotically in a certain region, and that it is possible to construct quasimodes associated with high frequencies that give rise to other kinds of vibrations. This is the case, for instance, with spectral stiff problems [LP97] or vibrating systems with a single concentrated mass (see [GLP99] and [Per03]).

Nevertheless, when the low frequencies converge toward the same positive value μ (see [LP03], [Per05], and [Per07]), it can be difficult to describe the asymptotic behavior, as $\varepsilon \to 0$, of the individual eigenfunctions and to obtain asymptotics for the first eigenfunction. In some of these problems, quasimodes $\widetilde{u}^\varepsilon$ that provide an approach to linear combinations of eigenfunctions associated with all the eigenvalues in intervals $[\mu - \delta^\varepsilon, \mu + \delta^\varepsilon]$, with $\delta^\varepsilon \to 0$ as $\varepsilon \to 0$, can be constructed. These quasimodes could concentrate asymptotically their support or their energy at points or thin layers. This happens, for example, when describing vibrations of systems with many concentrated masses near the boundary (see [LP03], [Per05], and [Per06]), or in models from geophysics ([BI06] and [Per07]).

For these vibrating systems, given the quasimode as an initial data, the solution of the evolution problem behaves as a standing wave affecting only small regions for a long period of time, which we determine in this chapter. Here we prove that, when considering the evolution problem (21.15), for a long time, namely, for $t \in [0, O((\delta^\varepsilon)^{-\delta})]$ with some positive δ, the solutions of (21.15) are approached through standing waves of the type $e^{\mathrm{i}\sqrt{\mu}t}\widetilde{u}^\varepsilon$. It turns out that the results hold for any eigenvalue μ of the local problem (21.4).

We emphasize that the results in this chapter can be stated in a more general abstract framework and extend to low-frequency and high-frequency vibrations of other vibrating systems (see [LP01], [LP03], [Per03], [Per04], and [Per06]). We also note that these results are very different from those in [LP93] and [LP95b], where the evolution problem (21.15) is used to derive results on spectral convergence for low frequencies, which are much weaker than those in Theorem 2.

Section 21.2 contains preliminary results on quasimodes for problem (21.3). Proofs of these results can be found in [Per04], [Per05], and [Per06]. Section 21.3 contains new results on the ε-dependent evolution problem (21.15) (see Theorem 3 and Remark 1). For brevity, we sketch only an idea of the proof.

21.2 The Spectral Problem

Let $\mathcal{A} : \mathbf{H} \longrightarrow \mathbf{H}$ be a linear, self-adjoint, positive, and compact operator on a separable Hilbert space $\mathbf{H}$. Let $\{\lambda_i^{-1}\}_{i=1}^\infty$ be the set of positive eigenvalues with the usual convention for repeated eigenvalues, $\lambda_i \to \infty$ as $i \to \infty$. Let $\{u_i\}_{i=1}^\infty$ be the set of eigenfunctions, which form an orthonormal basis for $\mathbf{H}$.

A *quasimode with remainder* $r > 0$ *for the operator* $\mathcal{A}$ is a pair $(u, \mu) \in \mathbf{H} \times \mathbb{R}$, with $||u||_H = 1$ and $\mu > 0$, such that $||\mathcal{A}u - \mu u||_{\mathbf{H}} \leq r$. If there is no ambiguity, u is also referred to as a quasimode.

The following result establishes the closeness in the space $\mathbf{H} \times \mathbb{R}$ of the eigenelements of the operator $\mathcal{A}$ to a given quasimode of $\mathcal{A}$ (see, for example, [OSY92] for a proof, and [Laz99] for a more general statement).

Theorem 1. *Given a quasimode (u, μ) with remainder r for $\mathcal{A}$, in each interval $[\mu - r^*, \mu + r^*]$ containing $[\mu - r, \mu + r]$, there are eigenvalues of the operator $\mathcal{A}$, $\{\lambda^{-1}_{i(r^*)+k}\}_{k=1,2,\dots,I(r^*)}$ for some index $i(r^*)$ and number $I(r^*) \geq 1$. In addition, there is $u^* \in \mathbf{H}$, $\|u^*\|_{\mathbf{H}} = 1$, $u^* = \sum_{k=1}^{I(r^*)} \alpha_k u_{i(r^*)+k}$ for certain constants α_k, and satisfying*

$$\|u - u^*\|_{\mathbf{H}} = \|u - \sum_{k=1}^{I(r^*)} \alpha_k u_{i(r^*)+k}\|_{\mathbf{H}} \leq \frac{2r}{r^*}. \tag{21.1}$$

From the literature on spectral perturbation problems it appears that, when Theorem 1 is applied, the spaces and operators under consideration often depend on a small parameter ε that converges to 0. Also, the function u and the numbers λ and r arising in the definition of a quasimode depend on this parameter. This is the case of the operators associated with vibrating systems with concentrated masses, namely, problem (21.3), which we study in this chapter. For the sake of completeness, in Subsection 21.2.1 we gather certain results on quasimodes for this problem (21.3), used subsequently in the proof of the results in Section 21.3.

21.2.1 The Spectral Perturbation Problem

Let Ω be a bounded domain of $\mathbb{R}^n$, $n = 2, 3$, with a Lipschitz boundary $\partial\Omega$. Let Σ and Γ_Ω be nonempty parts of the boundary such that $\partial\Omega = \bar{\Sigma} \cup \bar{\Gamma}_\Omega$; Σ is assumed to be in contact with $\{x_n = 0\}$. Let ε and η be two small parameters such that $\varepsilon \ll \eta$ and $\eta = \eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

For $n = 2$, let B be the semicircle $B = \{(y_1, y_2) \ / \ y_1^2 + y_2^2 < 1, y_2 < 0\}$ in the auxiliary space $\mathbb{R}^2$ with coordinates y_1, y_2. For $n = 3$, let B be the half-ball $B = \{(y_1, y_2, y_3) \ / \ y_1^2 + y_2^2 + y_3^2 < 1, y_3 < 0\}$ in the auxiliary space $\mathbb{R}^3$ with coordinates y_1, y_2, y_3. Let ∂B be the boundary of B, $\partial B = \bar{T} \cup \bar{\Gamma}$, where T is the part lying on $\{y_n = 0\}$. Let B^ε (and, similarly, T^ε, Γ^ε) denote its homothetic εB (εT, $\varepsilon \Gamma$). Let B_k^ε (and, similarly, T_k^ε, Γ_k^ε) denote the domain obtained by the translation of the previous B^ε (T^ε, Γ^ε) centered at the point $\tilde{x}_k$ of Σ at distance η between them. Here k is a parameter ranging from 1 to $N(\varepsilon)$, $k \in \mathbf{N}$. $N(\varepsilon)$ denotes the number of B_k^ε contained in Ω; $N(\varepsilon)$ is $O(\eta^{-1})$ for $n = 2$ and $O(\eta^{-2})$ for $n = 3$. The parameter α denotes the value

$$\alpha = \lim_{\varepsilon \to 0} \frac{-1}{\eta \ln \varepsilon} \text{ for } n = 2 \text{ and } \alpha = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\eta^2} \text{ for } n = 3. \tag{21.2}$$

We consider the eigenvalue problem

$$\begin{cases} -\Delta u^\varepsilon = \rho^\varepsilon \lambda^\varepsilon u^\varepsilon & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \Gamma_\Omega \cup \bigcup T^\varepsilon, \\ \frac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \Sigma - \overline{\bigcup T^\varepsilon}, \end{cases} \tag{21.3}$$

where $\rho^\varepsilon = \rho^\varepsilon(x)$ is the density function defined by

$$\rho^\varepsilon(x) = \frac{1}{\varepsilon^m} \text{ if } x \in \bigcup B^\varepsilon, \quad \rho^\varepsilon(x) = 1 \text{ if } x \in \Omega - \overline{\bigcup B^\varepsilon},$$

the symbol $\bigcup$ is extended, for fixed ε, to all the regions B_k^ε contained in Ω, and the parameter m is a real number, $m > 2$ (see [LP93]–[LP95b] for different values of the parameter m, boundary conditions, and domain shapes).

As is well known, problem (21.3) has a discrete spectrum. For fixed ε, let $\{\lambda_i^\varepsilon\}_{i=1}^\infty$ be the sequence of eigenvalues of (21.3), converging to ∞, with the classical convention for repeated eigenvalues. It has been proved (see [LP93]–[LP95b]) that they satisfy the estimates $C\varepsilon^{m-2} \leq \lambda_i^\varepsilon \leq C_i\varepsilon^{m-2}$, where C is a constant independent of ε and i and C_i is a constant independent of ε. Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the corresponding sequence of eigenfunctions, which is an orthogonal basis for the space $\mathbf{V}^\varepsilon$, where $\mathbf{V}^\varepsilon$ is the completion of $\{u \in \mathcal{D}(\bar{\Omega}) \,/\, u = 0 \text{ on } \Gamma_\Omega \cup \bigcup T^\varepsilon\}$ in the topology of $H^1(\Omega)$.

Convergence results for *the low frequencies*, the eigenvalues of order $O(\varepsilon^{m-2})$ of (21.3), and the associated eigenfunctions can be found in [Per04], [Per05], and [Per06]. Also, the limit behavior of the eigenelements for sequences of eigenvalues of order $O(1)$, the so-called *high frequencies*, is in [LP93], [LP95a], [LP95b], and [LP01]. As in the case of a single concentrated mass, in general, low frequencies are associated with *the local vibrations* of the concentrated masses, each one independent of the others. We have found only one exception: For $n = 3$ and $\alpha > 0$, these frequencies also give rise to *global vibrations* affecting the whole structure ([LP03]). Apart from this exception, the low frequencies and the corresponding eigenfunctions are asymptotically described, in a certain way, by a so-called *local eigenvalue problem* (21.4).

The local problem is posed in $\mathbb{R}^{n-} = \{y \in \mathbb{R}^n \,/\, y_n < 0\}$ as follows:

$$\left\{\begin{array}{rl} -\Delta_y U &= \lambda U \text{ in } B, \\ -\Delta_y U &= 0 \text{ in } \mathbb{R}^{n-} - \bar{B}, \\ [U] &= [\frac{\partial U}{\partial n_y}] = 0 \text{ on } \Gamma, \\ U = 0 \text{ on } T & , \ \frac{\partial U}{\partial y_n} = 0 \text{ on } \{y_n = 0\} - \bar{T}, \\ U(y) \to c, \text{ as } |y| \to \infty & , \ y_n < 0 \text{ when } n = 2, \\ U(y) \to 0, \text{ as } |y| \to \infty & , \ y_n < 0 \text{ when } n = 3, \end{array}\right. \tag{21.4}$$

where the brackets denote the jump across Γ, $\vec{n}_y$ is the unit outward normal to Γ, and c is some unknown but well-defined constant. The variable y is the *local variable*:

$$y = \frac{x - \tilde{x}_k}{\varepsilon}.$$

Problem (21.4) can be written as a standard eigenvalue problem with a discrete spectrum in the space $\widetilde{\mathcal{V}}$, where $\widetilde{\mathcal{V}}$ is the completion of $\{U \in \mathcal{D}(\overline{\mathbb{R}^{n-}}) \,/\, U = 0 \text{ on } T\}$ in the Dirichlet norm $\|\nabla_y U\|_{L^2(\mathbb{R}^{n-})}$ (see [LP93] and [LP95b]).

Theorem 2 allows us to assert that there are at least $l_0 N(\varepsilon)$ values $\lambda^\varepsilon_{i(\varepsilon)}/\varepsilon^{m-2}$ converging to each eigenvalue λ^0 of (21.4), with l_0 being the multiplicity of λ^0. The corresponding eigenfunctions U^ε [cf. (21.5)] are approached in the space $\widetilde{\mathbf{V}}^\varepsilon$ by the eigenfunctions of (21.4) associated with λ^0, concentrating their support asymptotically in neighborhoods of the concentrated masses as stated in Theorem 2.

Also, the results in Theorem 2 along with the results on comparison of spectra for perturbed domains in [Per05] allow us to obtain an important difference for the asymptotic behavior of the low frequencies between the dimensions $n = 2$ and $n = 3$ of the space. Namely, for $n = 2$, and for each $i = 1, 2, \ldots,$ $\lambda^\varepsilon_i/\varepsilon^{m-2} \to \lambda^0_1$, as $\varepsilon \to 0$, where λ^0_1 is the first eigenvalue of (21.4). This does not hold for $n = 3$ and the value of α in (21.2) strictly positive (see [Per05] and [Per06] for further explanations). Let us refer to [GLP99], [OSY92], and [SS89] to compare with the stronger results on the approach for the eigenfunctions in the case of a single concentrated mass, the case where the convergence of the rescaled spectrum of (21.3) to that of (21.4) with conservation of multiplicity holds.

Let us change the variable in (21.3) by setting $y = x/\varepsilon$. We obtain:

$$\int_{\Omega_\varepsilon} \nabla_y U^\varepsilon . \nabla_y V^\varepsilon \, dy = \frac{\lambda^\varepsilon}{\varepsilon^{m-2}} \int_{\Omega_\varepsilon} \beta^\varepsilon(y) U^\varepsilon V^\varepsilon \, dy, \quad \forall V^\varepsilon \in \widetilde{\mathbf{V}}^\varepsilon, \tag{21.5}$$

with Ω_ε being the domain $\{ y \, / \, \varepsilon y \in \Omega \}$ and $\beta^\varepsilon(y)$ in (21.5) defined as

$$\beta^\varepsilon(y) = 1 \text{ if } y \in \bigcup \tau_y B^\varepsilon, \qquad \text{and } \beta^\varepsilon(y) = \varepsilon^m \text{ if } y \in \Omega_\varepsilon - \overline{\bigcup \tau_y B^\varepsilon}, \tag{21.6}$$

where $\tau_y B^\varepsilon$ denote the transformed domains of the regions B^ε to the y variable. $\widetilde{\mathbf{V}}^\varepsilon$ is the functional space $\{U = U(y) \, / \, U(\varepsilon y) \in \mathbf{V}^\varepsilon\}$. We assume that the eigenfunctions $\{U^\varepsilon_i\}_{i=1}^\infty$, in the local variable, satisfy $\|U^\varepsilon\|_{\widetilde{\mathbf{V}}^\varepsilon} = 1$.

Let us introduce the self-adjoint positive and compact operator $\mathcal{A}^\varepsilon$ on $\widetilde{\mathbf{V}}^\varepsilon$, $\mathcal{A}^\varepsilon$ defined by the right-hand side of (21.5), namely,

$$\langle \mathcal{A}^\varepsilon U, V \rangle_{\widetilde{\mathbf{V}}^\varepsilon} = \int_{\bigcup \tau_y B^\varepsilon} UV \, dy + \varepsilon^m \int_{\Omega_\varepsilon - \overline{\bigcup \tau_y B^\varepsilon}} UV \, dy, \; \forall U, V \in \widetilde{\mathbf{V}}^\varepsilon, \tag{21.7}$$

with eigenelements $\{(\varepsilon^{m-2}/\lambda^\varepsilon_i, U^\varepsilon_i)\}_{i=1}^\infty$.

Let us consider λ^0 an eigenvalue of (21.4) of multiplicity l_0, and let U^0_1, $U^0_2, \ldots, U^0_{l_0}$ be the corresponding eigenfunctions, orthogonal in $\widetilde{\mathcal{V}}$ and satisfying $\|\nabla_y U^0_i\|_{L^2(\mathbb{R}^n-)} = 1$.

Let us introduce a function $\widetilde{\varphi}^\varepsilon(y)$ that depends on n. For $n = 2$, we consider $R_\varepsilon = \sqrt{\frac{\varepsilon + \eta/4}{\varepsilon}}$ and define $\widetilde{\varphi}^\varepsilon(y) = 0$ if $|y| \geq R^2_\varepsilon$,

$$\widetilde{\varphi}^\varepsilon(y) = 1 \text{ if } |y| \leq R_\varepsilon, \;\; \widetilde{\varphi}^\varepsilon(y) = 1 - \frac{\ln |y| - \ln R_\varepsilon}{\ln R_\varepsilon} \text{ if } R_\varepsilon \leq |y| \leq R^2_\varepsilon. \tag{21.8}$$

For $n = 3$, we consider $\tilde{\varphi}^\varepsilon$ as a smooth function that takes the value 1 in the semi-ball of radius $((\varepsilon + \eta/8)/\varepsilon)$, $B((\varepsilon + \eta/8)/\varepsilon)$, and is 0 outside the semi-ball of radius $((\varepsilon + \eta/4)/\varepsilon)$, $B((\varepsilon + \eta/4)/\varepsilon)$:

$$\tilde{\varphi}^\varepsilon(y) = \varphi\left(2\frac{|\varepsilon y| - \varepsilon}{\eta}\right), \tag{21.9}$$

where $\varphi \in C^\infty[0,1]$, $0 \leq \varphi \leq 1$, $\varphi = 1$ in $[0, 1/4]$, and $Supp(\varphi) \subset [0, 1/2]$.

Obviously, the elements of $\tilde{\mathbf{V}}^\varepsilon$ extended by zero in $\mathbb{R}^{n-} - \overline{\Omega_\varepsilon}$ are elements of $\tilde{\mathcal{V}}$. Moreover, we have (see [LP93] and [LP95b]) that $U_p^0 \tilde{\varphi}^\varepsilon \in \tilde{\mathbf{V}}^\varepsilon$, and, as $\varepsilon \to 0$, $U_p^0 \tilde{\varphi}^\varepsilon \longrightarrow U_p^0$ in $\tilde{\mathcal{V}}$.

For each $k = 1, 2, \ldots, N(\varepsilon)$, $p = 1, 2, \ldots, l_0$, we introduce the function

$$Z_{k,p}^\varepsilon(y) = \frac{U_p^0(y - \frac{\tilde{x}_k}{\varepsilon})\tilde{\varphi}^\varepsilon(y - \frac{\tilde{x}_k}{\varepsilon})}{\|\nabla_y(U_p^0 \tilde{\varphi}^\varepsilon)\|_{L^2(\mathbb{R}^{n-})}}. \tag{21.10}$$

The following estimates hold (see [Per04] and [Per06] for a proof):

$$\|\mathcal{A}^\varepsilon Z_{k,p}^\varepsilon - \frac{1}{\lambda^0} Z_{k,p}^\varepsilon\|_{\tilde{\mathbf{V}}^\varepsilon} \leq o_\varepsilon, \quad \forall k, p, \tag{21.11}$$

where o_ε does not depend on k and p and tends to 0 as $\varepsilon \to 0$,

$$o_\varepsilon = C\left(\ln \frac{\varepsilon + \eta/4}{\varepsilon}\right)^{-1/2} \text{ for } n = 2, \tag{21.12}$$

$$o_\varepsilon = C \max\{\left(\frac{\varepsilon}{\eta}\right)^{1/2}, \varepsilon^{m-2}\} \text{ for } n = 3, \tag{21.13}$$

with the constant C independent of ε.

Theorem 2. *Let us consider λ^0 an eigenvalue of (21.4) of multiplicity l_0, and let $U_1^0, U_2^0, \ldots, U_{l_0}^0$ be the corresponding eigenfunctions, assumed to be orthonormal in $\tilde{\mathcal{V}}$. For any $K > 0$, there is $\varepsilon^*(K)$ such that, for $\varepsilon < \varepsilon^*(K)$, $K < l_0 N(\varepsilon)$, and the interval $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$ contains eigenvalues of (21.5), $\lambda_{i(\varepsilon)}^\varepsilon/\varepsilon^{m-2}$, with total multiplicity greater than, or equal to, K; d^ε is a certain sequence, $d^\varepsilon \to 0$ as $\varepsilon \to 0$ and the interval $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$ does not contain other eigenvalues of (21.4) different from λ^0.*

In addition, for any β such $0 < \beta < 1$, and for $d^\varepsilon = (o_\varepsilon)^\beta$, there are $l_0 N(\varepsilon)$ functions, $\{U_{k,p}^\varepsilon\}_{k=1,N(\varepsilon)}^{p=1,l_0}$, $U_{k,p}^\varepsilon \in \tilde{\mathbf{V}}^\varepsilon$, such that $\|U_{k,p}^\varepsilon\|_{\tilde{\mathbf{V}}^\varepsilon} = 1$, $U_{k,p}^\varepsilon$ belongs to the eigenspace associated with all the eigenvalues in $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$, and

$$\|U_{k,p}^\varepsilon - Z_{k,p}^\varepsilon\|_{\tilde{\mathbf{V}}^\varepsilon} \leq 2(o_\varepsilon)^{1-\beta}. \tag{21.14}$$

In (21.14), $o_\varepsilon(1)$ is given by (21.12) when $n = 2$ ((21.13) when $n = 3$), $Z_{k,p}^\varepsilon$ is defined by (21.10), and $\tilde{\varphi}^\varepsilon(y)$ is defined by (21.8) when $n = 2$ ((21.9) when $n = 3$). These functions, $\{U_{k,p}^\varepsilon\}_{k=1,N(\varepsilon)}^{p=1,l_0}$, satisfy the property that for any extracted subset of K functions $\{U_{j_1}^\varepsilon, U_{j_2}^\varepsilon, \ldots, U_{j_K}^\varepsilon\}$, they are linearly independent.

Let us observe that formula (21.11) shows that $(Z^{\varepsilon}_{k,p}, 1/\lambda^0)$ is a quasimode of remainder o_ε for the operator $\mathcal{A}^\varepsilon$ defined in (21.5)–(21.7). In the same way, according to (21.1), the width of the interval $d^\varepsilon = (o_\varepsilon)^\beta$ in Theorem 2 and the bound in (21.14) provide the closeness of these quasimodes and eigenlements of $\mathcal{A}^\varepsilon$. Theorem 2 has been proved in [Per04] (see also [Per06]) by applying Theorem 1 and results on *almost orthogonality* for the quasimodes.

21.3 The Evolution Problem

Let us consider the set of functional spaces $\widetilde{\mathbf{V}}^\varepsilon$ and $\mathbf{H}^\varepsilon$, where $\widetilde{\mathbf{V}}^\varepsilon$ is introduced in Subsection 21.2.1 with the norm $\|\nabla_y u\|_{L^2(\varepsilon^{-1}\Omega)}$ and $\mathbf{H}^\varepsilon = \{U(y)\,/U(\varepsilon y) \in L^2(\Omega)$ with the norm $\|(\beta^\varepsilon)^{1/2} u\|_{L^2(\varepsilon^{-1}\Omega)}$, β^ε being defined by (21.6). Let A^ε be the operator associated with the form on $\widetilde{\mathbf{V}}^\varepsilon$ arising on the left-hand side of (21.5). Let $(Z^{\varepsilon}_{k,p}, 1/\lambda^0)$ be the quasimodes constructed in Subsection 21.2.1, for $k = 1, 2, \ldots, N(\varepsilon)$, $p = 1, 2, \ldots, l_0$, from the eigenelement (λ^0, U^0_p) of the local problem (21.4).

Let us consider the hyperbolic problem associated with (21.5):

$$\left\{\begin{array}{r} \dfrac{d^2\mathbf{U}^\varepsilon}{dt^2} + A^\varepsilon \mathbf{U}^\varepsilon = 0 \\ \mathbf{U}^\varepsilon(0) = \varphi^\varepsilon \\ \dfrac{d\mathbf{U}^\varepsilon}{dt}(0) = \psi^\varepsilon \end{array}\right. \tag{21.15}$$

For initial data $(\varphi^\varepsilon, \psi^\varepsilon) \in \widetilde{\mathbf{V}}^\varepsilon \times \mathbf{H}^\varepsilon$, problem (21.15) has a unique solution, $\mathbf{U}^\varepsilon \in L^\infty(0, \infty, \widetilde{\mathbf{V}}^\varepsilon)$, $\frac{d\mathbf{U}^\varepsilon}{dt} \in L^\infty(0, \infty, \mathbf{H}^\varepsilon)$, satisfying $\mathbf{U}^\varepsilon(0) = \varphi^\varepsilon$, and, for any fixed $T > 0$,

$$\int_0^T \left(\int_{\varepsilon^{-1}\Omega} \nabla_y \mathbf{U}^\varepsilon . \nabla_y \mathbf{\Phi}\, dy - \int_{\varepsilon^{-1}\Omega} \beta^\varepsilon(y) \frac{d\mathbf{U}^\varepsilon}{dt} \frac{d\mathbf{\Phi}}{dt}\, dy \right) dt =$$

$$\int_{\varepsilon^{-1}\Omega} \beta^\varepsilon(y) \psi^\varepsilon \mathbf{\Phi}(0)\, dy$$

for any test function $\mathbf{\Phi}$ of the form $\mathbf{\Phi} = \phi(t) V$, where $V \in \widetilde{\mathbf{V}}^\varepsilon$, and $\phi \in C^1([0,T])\,/\,\phi(T) = 0$ (cf. [SS89] and [S80], for instance).

Because of the conservation of energy, for each $t \in R$ we have

$$\|\mathbf{U}^\varepsilon(t)\|_{\widetilde{\mathbf{V}}^\varepsilon} + \left\| \frac{d\mathbf{U}^\varepsilon}{dt}(t) \right\|_{\mathbf{H}^\varepsilon} = \|\varphi^\varepsilon\|_{\widetilde{\mathbf{V}}^\varepsilon} + \|\psi^\varepsilon\|_{\mathbf{H}^\varepsilon}. \tag{21.16}$$

According to the Fourier expansion of $\mathbf{U}^\varepsilon(t)$ in terms of the eigenfunctions of (21.5), for a given $\varphi^\varepsilon = aU^\varepsilon_{i(\varepsilon)}$ and $\psi^\varepsilon = bU^\varepsilon_{i(\varepsilon)}$, with a and b any constants and $U^\varepsilon_{i(\varepsilon)}$ any eigenfunction of (21.5) associated with the eigenvalue $\lambda^\varepsilon_{i(\varepsilon)}/\varepsilon^{m-2}$, the solution of (21.15) is the standing wave

$$\mathbf{U}^\varepsilon(t) = \left(a\,\cos\left(\sqrt{\frac{\lambda^\varepsilon_{i(\varepsilon)}}{\varepsilon^{m-2}}}\,t\right) + b\sqrt{\frac{\varepsilon^{m-2}}{\lambda^\varepsilon_{i(\varepsilon)}}}\,\sin\left(\sqrt{\frac{\lambda^\varepsilon_{i(\varepsilon)}}{\varepsilon^{m-2}}}\,t\right)\right) U^\varepsilon_{i(\varepsilon)}.$$

Similarly, for any given data, the functions $U^\varepsilon_{k,p}$ arising in Theorem 2, namely, $\varphi^\varepsilon = \sum_{j=1}^{l_0 N(\varepsilon)} a_j\, U^\varepsilon_{i(\varepsilon)+j}$ and $\psi^\varepsilon = \sum_{j=1}^{l_0 N(\varepsilon)} b_j\, U^\varepsilon_{i(\varepsilon)+k}$ with a_j and b_j constants, the solution of (21.15) is

$$\mathbf{U}^\varepsilon(t) = \sum_{j=1}^{l_0 N(\varepsilon)} \left(a_j \cos\left(\sqrt{\frac{\lambda^\varepsilon_{i(\varepsilon)+j}}{\varepsilon^{m-2}}}\,t\right)\right.$$
$$\left. + b_j\sqrt{\frac{\varepsilon^{m-2}}{\lambda^\varepsilon_{i(\varepsilon)+j}}}\,\sin\left(\sqrt{\frac{\lambda^\varepsilon_{i(\varepsilon)+j}}{\varepsilon^{m-2}}}\,t\right)\right) U^\varepsilon_{i(\varepsilon)+j}.$$

By contrast, in the case where the initial data are the quasimodes of the operator $\mathcal{A}^\varepsilon$ arising in Theorem 2, namely, the $Z^\varepsilon_{k,p}$ associated with the eigenelement (λ^0, U^0_p) of (21.4), for $k = 1, 2, \ldots, N(\varepsilon)$ and $p = 1, 2, \ldots, l_0$, approaching the functions $U^\varepsilon_{k,p}$ (see (21.14)), the solutions of the evolution problem (21.15) are not standing waves or sums of standing waves.

Nevertheless, the following theorem establishes the range of t where the standing wave $\cos(\sqrt{\lambda^0}\,t) Z^\varepsilon_{k,p}$ ($\sqrt{(\lambda^0)^{-1}}\sin(\sqrt{\lambda^0}\,t) Z^\varepsilon_{k,p}$, respectively) approaches the solution $\mathbf{U}^\varepsilon(t)$ of (21.15) for the initial data $(\varphi^\varepsilon, \psi^\varepsilon) = (Z^\varepsilon_{k,p}, 0)$ ($(\varphi^\varepsilon, \psi^\varepsilon) = (0, Z^\varepsilon_{k,p})$, respectively).

Theorem 3. *Let (λ^0, U^0_p) be an eigenelement of (21.4), and let $Z^\varepsilon_{k,p}$ be defined by (21.10) for $k = 1, 2, \ldots, N(\varepsilon)$ and $p = 1, 2, \ldots, l_0$. Let us consider problem (21.15) for $(\varphi^\varepsilon, \psi^\varepsilon) = (Z^\varepsilon_{k,p}, 0)$. Then, for $t > 0$ and sufficiently small ε (namely, $\varepsilon < \varepsilon_0$ with ε_0 independent of t), the unique solution $\mathbf{U}^\varepsilon(t)$ of (21.15) satisfies*

$$\left\|\cos(\sqrt{\lambda^0}\,t) Z^\varepsilon_{k,p} - \mathbf{U}^\varepsilon(t)\right\|_{\tilde{\mathbf{V}}^\varepsilon} \leq C_1 \max\left((o_\varepsilon)^{1-\beta}, (o_\varepsilon)^{\beta/2} t\right), \tag{21.17}$$

$$\left\|\sqrt{\lambda^0}\sin(\sqrt{\lambda^0}\,t) Z^\varepsilon_{k,p} + \frac{d\mathbf{U}^\varepsilon}{dt}(t)\right\|_{\mathbf{H}^\varepsilon} \leq C_2 \max\left((o_\varepsilon)^{1-\beta}, ((o_\varepsilon)^{\beta/2} t + (o_\varepsilon)^{\beta/2}\right), \tag{21.18}$$

where C_1 and C_2 are constants that may depend on λ^0 but are independent of ε and t, o_ε is defined by (21.12) when $n = 2$ and by (21.13) when $n = 3$, and β is the constant appearing in (21.14), $0 < \beta < 1$.

In the same way, for $(\varphi^\varepsilon, \psi^\varepsilon) = (0, Z^\varepsilon_{k,p})$, the following estimates hold:

$$\left\|\frac{\sin(\sqrt{\lambda^0}\,t)}{\sqrt{\lambda^0}} Z^\varepsilon_{k,p} - \mathbf{U}^\varepsilon(t)\right\|_{\tilde{\mathbf{V}}^\varepsilon} \leq C_1 \max\left((o_\varepsilon)^{1-\beta}, (o_\varepsilon)^{\beta/2} t\right), \tag{21.19}$$

$$\left\|\cos(\sqrt{\lambda^0}\,t) Z^\varepsilon_{k,p} - \frac{d\mathbf{U}^\varepsilon}{dt}(t)\right\|_{\mathbf{H}^\varepsilon} \leq C_2 \max\left((o_\varepsilon)^{1-\beta}, ((o_\varepsilon)^{\beta/2} t + (o_\varepsilon)^{\beta/2}\right). \tag{21.20}$$

The proof of Theorem 3 is based on (21.16), on the precise bounds (21.11)–(21.14), and on the inequality $\|u\|_{\mathbf{H}^\varepsilon} \leq C\|u\|_{\tilde{\mathbf{V}}^\varepsilon}$ $\forall u \in \mathbf{V}^\varepsilon$, where C is a constant independent of u and ε. For the sake of brevity, we omit the proof, which will be provided in a future publication.

Remark 1. In fact, approaches (21.17)–(21.20) in Theorem 3 hold uniformly for $t \in [0, (o_\varepsilon)^{-\beta\beta'/2}]$ for any constant β' satisfying $0 < \beta' < 1$. Then the bounds on the right-hand side of (21.17)–(21.20) are $C^*(o_\varepsilon)^{\min(1-\beta,\,\beta(1-\beta')/2)}$, where C^* is a constant independent of ε.

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