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Long time approximations for solutions of wave equations via standing waves from quasimodes ☆

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Abstract

A quasimode for a positive, symmetric and compact operator on a Hilbert space could be defined as a pair (u, λ), where u is a function approaching a certain linear combination of eigenfunctions associated with the eigenvalues of the operator in a "small interval" $[\lambda - r, \lambda + r]$. Its value in describing asymptotics for low and high frequency vibrations in certain singularly perturbed spectral problems, which depend on a small parameter ε, has been made clear recently in many papers. In this paper, considering second order evolution problems, we provide estimates for the time t in which *standing waves* of the type $e^{\mathrm{i}\sqrt{\lambda}t}u$ approach their solutions $\mathbf{u}(t)$ when the initial data deal with quasimodes of the associated operators. We establish a general abstract framework and we extended it to the case where operators and spaces depend on the small parameter ε: now λ and u can depend on ε and also perform the estimates for t. We apply the results to vibrating systems with concentrated masses.

Résumé

On peut définir un « quasimode » pour un opérateur positif, symétrique, compact sur un espace de Hilbert comme un couple (u, λ), où u est une fonction approchant une certaine combinaison linéaire de fonctions propres associées à des valeurs propres de l'opérateur appartenant à un « petit intervalle » $[\lambda - r, \lambda + r]$. Récemment dans plusieurs articles on a mis en évidence l'intérêt des quasimodes pour décrire le comportement asymptotique des vibrations de basses et de hautes fréquences dans certains problèmes spectraux, singulièrement perturbés, dépendant d'un petit paramètre ε qui tend vers 0. Dans cet article on considère des problèmes d'évolution du deuxième ordre ; on obtient des estimations précises pour le temps t pendant lequel les *ondes stationnaires* du type $e^{\mathrm{i}\sqrt{\lambda}t}u$ donnent des approximations des solutions $\mathbf{u}(t)$ de problèmes d'évolution quand les données initiales sont liées aux quasimodes des opérateurs associés. On établit un cadre abstrait très général que l'on étend au cas où les opérateurs et les espaces dépendent du petit paramètre ε : ici λ et u peuvent dépendre de ε ainsi que les estimations en t. On applique les résultats à des systèmes vibrant avec des masses concentrées.

Keywords: Spectral analysis; Asymptotic analysis; Quasimodes; Standing waves

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1. Introduction

In this paper, we consider vibrating systems for which the associated spectral problems are singularly perturbed problems which depend on a small parameter ε. The abstract framework for these problems can be established as follows:

Let us consider a positive, self-adjoint, compact operator $\mathbf{A}^\varepsilon$ on a Hilbert space $\mathbf{H}^\varepsilon$, ε being a parameter that converge towards zero. For fixed ε, let $\{(\mu_i^\varepsilon)^{-1}\}_{i=1}^\infty$ be the set of eigenvalues of $\mathbf{A}^\varepsilon$, $\mu_i^\varepsilon \to \infty$ as $i \to \infty$. Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the associated eigenfunctions, which are assumed to form an orthonormal basis in $\mathbf{H}^\varepsilon$. Assuming that the low frequencies (or the re-scaled low frequencies) μ_i^ε, with a fixed i, are of order $O(1)$, the high frequencies are referred to as the sequences of eigenvalues $\mu_{i(\varepsilon)}^\varepsilon$ of order $O(\varepsilon^{-\alpha})$ for some $\alpha \geqslant 0$ and $i(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Considering standard vibration problems associated with these operators (within the suitable abstract framework), the elementary vibrations are the standing waves $\exp(\mathrm{i}\sqrt{\mu_j^\varepsilon}t)u_j^\varepsilon$.

On account of the difficulty of explicit and/or numeric computations of the eigenelements $(\mu_j^\varepsilon, u_j^\varepsilon)$ for singularly perturbed spectral problems, many papers in the literature deal with the asymptotics for these eigenelements. Nevertheless, there is a lack of studies of asymptotics for solutions of the associated time dependent problems when the initial data are in neighborhood of the eigenfunctions as $\varepsilon \to 0$. This is one of the main aims of this paper. As a matter of fact, we show that in the case where approaches to eigenfunctions individually are not available, then *approaches through quasimodes can work as approaches to eigenfunctions individually* (cf. Remark 4.4). Here *approaches* must be understood in the framework of the usual energy spaces.

Roughly speaking, a *quasimode* $\tilde{u}^\varepsilon$ for a spectral problem can be defined as a function approaching a certain linear combination of eigenfunctions associated with the eigenvalues in a "small interval" $[\tilde{\lambda}^\varepsilon - \delta^\varepsilon, \tilde{\lambda}^\varepsilon + \delta^\varepsilon]$. In fact, the definition of a quasimode is related with an operator A^ε on a Hilbert space $\mathbf{H}^\varepsilon$, the *almost frequency* $\tilde{\lambda}^\varepsilon$ and the *reminder* δ^ε (cf. Remark 2.6). Its value in describing asymptotics for high frequency vibrations in certain singularly perturbed spectral problems has been made clear recently in many papers. Let us refer, for instance, to [26,31–33] and [34] for vibrating systems with concentrated masses and further references on other spectral stiff problems and to [35] for boundary homogenization problems. But also for operators on Hilbert spaces which do not depend on perturbation parameters, the quasimodes are of interest in describing high frequency vibrations; see [2,4,15,31] in this connection.

We emphasize that for problems arising in spectral perturbation theory, constructing quasimodes can be important to describe low and high frequency vibrations in the case where obtaining approaches to true eigenfunctions of the original ε-dependent problem is difficult (see [6,11,12,14,18,19,23–26,31,32,34,35]). This happens, for instance, when consecutive eigenvalues are very close and we do not have any precise information on the distance between these eigenvalues. Below are some comments in this respect:

(i) As regards the high frequencies, under certain hypotheses on the operators, it is known that the re-scaled high frequencies $\mu_{i(\varepsilon)}^\varepsilon \varepsilon^\alpha$, with $\alpha > 0$, accumulate asymptotically on the whole positive real axis, being at a small distance between them, and it is difficult to study the asymptotic behavior of the associated eigenfunctions $u_{i(\varepsilon)}^\varepsilon$ individually (cf. for instance [8,12,23], [25] and [37]).
Besides, in certain problems the low frequencies converge, as $\varepsilon \to 0$, towards those of a limit problem with conservation of the multiplicity. If so, there is also a convergence for the associated eigenfunctions, but it may occur that these low frequencies give rise to vibrations of the system which are asymptotically concentrated in a certain region and it is possible to construct quasimodes associated with high frequencies giving rise to other kinds of vibrations. This is the case, for instance, of spectral stiff problems (cf. [27,18,19,23]) or vibrating systems with a single concentrated mass (cf. [12,16,29,30,38,31]) or with the mass concentrated along a curve (cf. [11]).

(ii) For the low frequencies, in the case where they converge towards the same positive value μ; that is, for fixed $i = 1, 2, \ldots$, $\mu_i^\varepsilon \to \mu > 0$ as $\varepsilon \to 0$ (cf. [26,33] and [35]), it can be difficult to describe the asymptotic behavior as $\varepsilon \to 0$ of the eigenfunctions u_i^ε individually, or even asymptotics for the first eigenfunction u_1^ε. This fact clearly depends on the normalization of the eigenfunctions.
Sometimes quasimodes $\tilde{u}^\varepsilon$ providing an approach to linear combinations of eigenfunctions associated with all the eigenvalues in intervals $[\mu - \delta^\varepsilon, \mu + \delta^\varepsilon]$, with $\delta^\varepsilon \to 0$ as $\varepsilon \to 0$, can be constructed. These quasimodes

could concentrate asymptotically their support and/or their energy at points or thin layers. This is the case of vibrating systems with many concentrated masses near the boundary (cf. [24,26,32,34]) or in models from Geophysics (cf. [5,7,35]) and in reinforcement problems depending on the geometry of the reinforced region (see [13]).

For all these vibrating systems described above, given a quasimode as an initial data, we show that the solution of the evolution problem behaves as a standing wave for a long time, a time which we establish in this paper (cf. Remarks 3.3, 4.1 and 4.5). This makes it possible to detect standing waves affecting only certain regions, for long periods of time. Specifying, we prove here that for a long time, namely $t \in [0, O((\delta^\varepsilon)^{-\beta})]$, for some positive β, the solution of the associated evolution problem is approached by functions of the type $\exp(\mathrm{i}\sqrt{\mu}t)\tilde{u}^\varepsilon$ (or $\exp(\mathrm{i}\sqrt{\tilde{\lambda}^\varepsilon}t)\tilde{u}^\varepsilon$). These functions (namely, their real or imaginary parts) shall be henceforth referred to as *standing waves*, though they may not be solutions of wave equations (cf. Remarks 2.3 and 2.4). In this framework, we note that this is not far from giving approaches to eigenfunctions individually, since these approaches, when taken as initial data, also originate standing waves for long times (cf. Remark 4.4). The aim of Section 4 is to present a sample where the behaviors described in (i) and (ii) for the asymptotics of the eigenvalues occur: accumulation of the high frequencies on the whole positive real axis and concentration of the low frequencies at a point.

Finally, it should be noted that the idea in this paper is different from the idea used in previous works where the convergence of the solutions of the evolution problems for certain initial data, by means of the Fourier or Laplace transforms, provide a convergence of the associated spectral families which leads to a certain spectral convergence for low and high frequencies (see [23,38,39] as general references and Remarks 4.3 and 4.10 to compare). Here we use stronger convergence, from results in previous works (cf. [25] and [26]), to obtain approaches for solutions of evolution problems via standing waves (see, for instance, Remarks 4.1, 4.3, and 4.5).

The structure of the paper is as follows: Section 2 contains the general abstract framework for evolution problems and quasimodes for operators and Hilbert spaces which do not depend on the perturbation parameter. The more general results are given by Theorems 2.2 and 2.3 and Corollaries 2.1 and 2.2 (see also Remarks 2.4 and 2.5). They allow a wide range of application when considering problems from the spectral perturbation theory (cf. also Remark 3.2). Section 3 contains an abstract framework for problems arising in spectral perturbation theory. The above mentioned frameworks are general for standard vibration problems (cf. Sections I–III in [38], for instance): the associated semigroups of contraction are unitary and the conservation of the energy is used to derive uniform bounds for the discrepancies between solutions of the evolution problems and standing waves.

In Section 4 we apply the results in Sections 2 and 3 to vibrating systems with concentrated masses which have been widely studied in the literature. Namely, we consider the vibrations of a body occupying a domain Ω of $\mathbb{R}^n$, $n = 2, 3$, that contains many small regions B^ε of high density near the boundary, the so-called *concentrated masses*. The small parameter ε deals with the size of the masses, their number and their densities. The asymptotic behavior, as $\varepsilon \to 0$, for the eigenelements $(\lambda^\varepsilon, u^\varepsilon)$ of the corresponding spectral problem (4.2) has been treated in [32] when $\lambda^\varepsilon = O(\varepsilon^{m-2})$ and in [25] when $\lambda^\varepsilon = O(1)$. We refer to [24,25,31–34], as previous works where quasimodes that we use throughout Section 4 have been constructed and to [26] for a large list of references on the low and high frequencies for this problem. Here, considering the second order evolution problem (4.5), we provide estimates for the time t in which certain standing waves approach time dependent solutions when the initial data are quasimodes. Also precise bounds for the discrepancies between the solutions and the standing waves in the suitable Sobolev spaces are provided (see Theorems 4.3, 4.4, and 4.6). Section 4.2 contains the new results related to the low frequency vibrations while Section 4.3 has those related to the high frequency vibrations. It should be noted that a re-scaling of the eigenvalues of the original ε-dependent spectral problem leads to a re-scaling of times and velocities in terms of the solutions of the wave equations. The phenomena also recalls that noticed in [40] on the different time scales of observation of disturbances depending on the initial disturbance of the media (see (4.21)–(4.24), (4.26)–(4.27), (4.37)–(4.41) and Remarks 4.1, 4.5, and 4.6).

We note that some of the results in Section 4.2 have been announced without any proof in [36] while we provide here their proofs using the results in Section 3. See [11,12,18,19,23,31,34,35], for other problems of spectral perturbation theory where we can apply the general results of Section 3 and see [26] for a large bibliography. Finally, it should be mentioned that comments on extensions of the technique in this paper to other spectral perturbed problems and comparisons with different techniques and results in previous papers can be found in the introduction of Sections 4 and 4.1 as well as in Remarks 4.2–4.11.

2. The general abstract framework

Let $\mathcal{A}: \mathbf{H} \to \mathbf{H}$ be a linear, self-adjoint, positive and compact operator on a separable Hilbert space $\mathbf{H}$. Let $\{\lambda_i^{-1}\}_{i=1}^{\infty}$ be the set of positive eigenvalues with the usual convention of repeated eigenvalues, $\lambda_i \to \infty$ as $i \to \infty$. Let $\{u_i\}_{i=1}^{\infty}$ be the set of eigenfunctions which form an orthonormal basis of $\mathbf{H}$.

A *quasimode with remainder* $r > 0$ for the operator $\mathcal{A}$ is a pair $(u, \mu) \in \mathbf{H} \times \mathbb{R}$, with $\|u\|_{\mathbf{H}} = 1$ and $\mu > 0$, such that $\|\mathcal{A}u - \mu u\|_{\mathbf{H}} \leqslant r$. Throughout the text, if no confusion arise, we shall refer to u as the *quasimode*, which is associated with the *almost frequency* μ and the rest r.

The following theorem establishes the closeness in the space $\mathbf{H} \times \mathbb{R}$ of the eigenelements of the operator $\mathcal{A}$ to a given quasimode of $\mathcal{A}$ (see [41] for the proof).

Theorem 2.1. *Let $\mathcal{A}: H \to H$ be a linear, self-adjoint, positive and compact operator on a separable Hilbert space H. Let $u \in H$, with $\|u\|_H = 1$ and $\mu, r > 0$ such that $\|\mathcal{A}u - \mu u\|_H \leqslant r$. Then, there exists an eigenvalue μ_i of $\mathcal{A}$ satisfying $|\lambda - \mu_i| \leqslant r$. Moreover, for any $r^* > r$ there is $u^* \in H$, with $\|u^*\|_H = 1$, u^* belonging to the eigenspace associated with all the eigenvalues of the operator $\mathcal{A}$ lying on the segment $[\mu - r^*, \mu + r^*]$ and such that*

$$\|u - u^*\|_H \leqslant \frac{2r}{r^*}.$$

We refer to [15] for the definition of families of quasimodes, results on the total multiplicity of the eigenvalues in the interval $[\mu - r, \mu + r]$ and a more general statement of Theorem 2.1 (in the case where $\mathcal{A}$ is not necessarily a positive operator). In fact, here Theorem 2.1 can be rewritten as follows:

Given a quasimode (u, μ) for the operator $\mathcal{A}$ of remainder r, in each interval $[\mu - r^*, \mu + r^*]$ containing $[\mu - r, \mu + r]$ there are eigenvalues of $\mathcal{A}$, $\{\lambda^{-1}_{i(r^*)+k}\}_{k=1,2,\cdots,I(r^*)}$ for some index $i(r^*)$ and some natural number $I(r^*) \geqslant 1$. In addition, there is $u^* \in \mathbf{H}$,

$$\|u^*\|_{\mathbf{H}} = 1, \quad u^* \in [u_{i(r^*)+1}, u_{i(r^*)+2}, \cdots, u_{i(r^*)+I(r^*)}], \quad u^* = \sum_{k=1}^{I(r^*)} \alpha_k u_{i(r^*)+k}, \tag{2.1}$$

satisfying

$$\|u - u^*\|_{\mathbf{H}} = \left\| u - \sum_{k=1}^{I(r^*)} \alpha_k u_{i(r^*)+k} \right\|_{\mathbf{H}} \leqslant \frac{2r}{r^*}, \tag{2.2}$$

where, on account of the normalization for the eigenfunctions and u^*, the α_k are constants such that $|\alpha_k| \leqslant 1$ for $k = 1, 2, \ldots I(r^*)$.

The following theorem shows the connection between a quasimode of an operator and the standing waves of the type $e^{\mathrm{i}\sqrt{\mu}t}u$ approaching solutions of second order evolution problems. In order to prove the theorem, let us introduce a general abstract framework of the standard vibration problem for a system with discrete spectrum.

Let $\mathbf{V}$ and $\mathbf{H}$ be two separable Hilbert spaces, $\mathbf{V} \subset \mathbf{H}$, with a dense and compact imbedding. Let $a(u, v)$ be a sesquilinear, hermitian, continuous and coercive form on $\mathbf{V}$. We consider $\mathbf{V}$ equipped with the scalar product inducted by $a(\cdot,\cdot)$. Let $A \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$ be the associated operator with the form a, namely, $a(u, v) = \langle Au, v\rangle_{\mathbf{V}' \times \mathbf{V}}$, and, let us consider the associated spectral problem: to find λ and $u \in \mathbf{V}$, $u \neq 0$ satisfying,

$$a(u, v) = \lambda(u, v)_{\mathbf{H}}, \quad \forall v \in \mathbf{V}.$$

Let A_H be the operator restriction of A to $\mathbf{H}$, that is the restriction of the operator A to the domain of definition $D(A_H) = \{v \in \mathbf{V} / Av \in \mathbf{H}\}$, defined by $A_h f = u_f$ where u_f is the solution of $a(u_f, v) = (f, v)_{\mathbf{H}}$. Then, $\mathcal{A} = A_H^{-1}$, $\mathcal{A}: \mathbf{H} \to \mathbf{H}$ is an operator which satisfies the properties in Theorem 2.1. The eigenvalues of A (respectively $\mathcal{A}$) are $\{\lambda_i\}_{i=1}^{\infty}$ (respectively $\{\lambda_i^{-1}\}_{i=1}^{\infty}$), and the associated eigenfunctions are $\{u_i\}_{i=1}^{\infty}$ which form an orthogonal basis in $\mathbf{H}$ and $\mathbf{V}$, of norm 1 in $\mathbf{H}$ and of norm $\sqrt{\lambda_i}$ in $\mathbf{V}$.

Let us consider the evolution problem,

$$\begin{cases} \dfrac{d^2\mathbf{u}}{dt^2} + A\mathbf{u} = 0, \\ \mathbf{u}(0) = \varphi, \\ \dfrac{d\mathbf{u}}{dt}(0) = \psi. \end{cases} \tag{2.3}$$

For initial data $(\varphi, \psi) \in \mathbf{V} \times \mathbf{H}$, there is a unique solution $\mathbf{u}(t)$ of problem (2.3) which satisfies:

$$\mathbf{u} \in L^\infty(0, \infty, \mathbf{V}), \quad \frac{d\mathbf{u}}{dt} \in L^\infty(0, \infty, \mathbf{H}), \tag{2.4}$$

$$\mathbf{u}(0) = \varphi, \tag{2.5}$$

and, for any fixed $T > 0$,

$$\int_0^T \left(a\big(\mathbf{u}(t), \boldsymbol{\Phi}(t)\big) - \left\langle \frac{d\mathbf{u}}{dt}, \frac{d\boldsymbol{\Phi}}{dt} \right\rangle_{\mathbf{H}} \right) dt = \big\langle \psi, \boldsymbol{\Phi}(0) \big\rangle_{\mathbf{H}} \tag{2.6}$$

for any test function $\boldsymbol{\Phi}$ of the form $\boldsymbol{\Phi}(t) = \phi(t)v$, where v is any element in a dense set of $\mathbf{V}$ and ϕ is any function in $\{\phi \in C^1([0, T])/\phi(T) = 0\}$.

The solution $\mathbf{u}(t)$ of (2.3) has the Fourier expansion,

$$\mathbf{u}(t) = \sum_{i=1}^{\infty} \varphi_i \cos(\sqrt{\lambda_i} t) u_i + \frac{\psi_i}{\sqrt{\lambda_i}} \sin(\sqrt{\lambda_i} t) u_i, \tag{2.7}$$

where φ_i and ψ_i are the Fourier coefficients of the initial data, namely,

$$\varphi = \sum_{i=1}^{\infty} \varphi_i u_i \quad \text{in } \mathbf{V}, \qquad \psi = \sum_{i=1}^{\infty} \psi_i u_i \quad \text{in } \mathbf{H},$$

and the conservation of the energy is satisfied:

$$a\big(\mathbf{u}(t), \mathbf{u}(t)\big) + \left\| \frac{d\mathbf{u}}{dt}(t) \right\|_{\mathbf{H}}^2 = a(\varphi, \varphi) + \|\psi\|_{\mathbf{H}}^2 = \sum_{i=1}^{\infty} \sqrt{\lambda_i^2 \varphi_i^2} + \sum_{i=1}^{\infty} \psi_i^2, \quad \forall t \in \mathbb{R} \tag{2.8}$$

(see Section III.8 in [17], Sections I.6 and III.11 in [38], Sections IV.5 and XII.3 in [39]).

It is self-evident that, from (2.7), for a given $\varphi = \alpha u_i$ or $\psi = \beta u_i$, with α, β any constants, u_i any eigenfunction of A associated with the eigenvalue λ_i, the solution of (2.3) is the standing wave,

$$\mathbf{u}(t) = \left(\alpha \cos(\sqrt{\lambda_i} t) + \beta \frac{\sin(\sqrt{\lambda_i} t)}{\sqrt{\lambda_i}} \right) u_i.$$

Similarly, for

$$\varphi = \sum_{k=1}^{I(r^*)} a_k u_{i(r^*)+k} \quad \text{and} \quad \psi = \sum_{k=1}^{I(r^*)} b_k u_{i(r^*)+k}, \tag{2.9}$$

with a_k, b_k constants, the solution of (2.3) is given by

$$\mathbf{u}(t) = \sum_{k=1}^{I(r^*)} \left(a_k \cos(\sqrt{\lambda_{i(r^*)+k}}\, t) + b_k \frac{\sin(\sqrt{\lambda_{i(r^*)+k}}\, t)}{\sqrt{\lambda_{i(r^*)+k}}} \right) u_{i(r^*)+k}. \tag{2.10}$$

Also, for solutions of problem (2.3),

$$\mathbf{u}(t) = \sum_{k=1}^{I(r^*)} a_k \big(\cos(\sqrt{\lambda_{i(r^*)+k}}\, t) + \sin(\sqrt{\lambda_{i(r^*)+k}}\, t) \big) u_{i(r^*)+k}, \tag{2.11}$$

associated with complex solutions of the form $\mathbf{u}(t)=\sum_{k=1}^{I(r^*)} a_k e^{i\sqrt{\lambda_{i(r^*)+k}}\,t} u_{i(r^*)+k}$, the associated initial data are:

$$\varphi=\sum_{k=1}^{I(r^*)} a_k u_{i(r^*)+k}, \qquad \psi=\sum_{k=1}^{I(r^*)} a_k\sqrt{\lambda_{i(r^*)+k}}\,u_{i(r^*)+k}. \tag{2.12}$$

Theorem 2.2. *Let* (u,λ^{-1}) *be a quasimode with remainder* r *of the operator* $\mathcal{A}=A_H^{-1}$, A *arising in* (2.3). *Let* $\{\lambda^{-1}_{i(r^*)+k}\}_{k=1,2,\dots,I(r^*)}$ *and* $\{u_{i(r^*)+k}\}_{k=1,2,\dots,I(r^*)}$ *be the eigenvalues and the associated eigenfunctions of the operator* $\mathcal{A}$ *satisfying* (2.1)–(2.2). *Let us assume that* $r^*>r$ *and* $\lambda^{-1}-r^*>0$. *Then, for* $\varphi=0$, $\psi=u$, *the solution* $\mathbf{u}(t)$ *of* (2.3) *satisfies*

$$a\left(\mathbf{u}(t)-\sum_{k=1}^{I(r^*)}\alpha_k\frac{\sin(\sqrt{\lambda_{i(r^*)+k}}\,t)}{\sqrt{\lambda_{i(r^*)+k}}}u_{i(r^*)+k},\ \mathbf{u}(t)-\sum_{k=1}^{I(r^*)}\alpha_k\frac{\sin(\sqrt{\lambda_{i(r^*)+k}}\,t)}{\sqrt{\lambda_{i(r^*)+k}}}u_{i(r^*)+k}\right)$$
$$+\left\|\frac{d\mathbf{u}}{dt}(t)-\sum_{k=1}^{I(r^*)}\alpha_k\cos(\sqrt{\lambda_{i(r^*)+k}}\,t)u_{i(r^*)+k}\right\|_{\mathbf{H}}^2\leqslant\left(\frac{2r}{r^*}\right)^2,\quad \forall t>0. \tag{2.13}$$

In addition, for any $t>0$, *we have:*

$$\left\|\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}u-\mathbf{u}(t)\right\|_{\mathbf{H}}$$
$$\leqslant 3\max\left(\frac{2rC_{\mathbf{V}}}{r^*},\frac{2r}{r^*\sqrt{\lambda}},C^*\left(\frac{1}{\sqrt{\lambda}}+\sqrt{r^*}\right)\max_{1\leqslant k\leqslant I(r^*)}\left|\sin(\sqrt{\lambda_{i(r^*)+k}}\,t)-\sin(\sqrt{\lambda}t)\right|+C^*\sqrt{r^*}\right), \tag{2.14}$$

and

$$\left\|\cos(\sqrt{\lambda}t)u-\frac{d\mathbf{u}}{dt}(t)\right\|_{\mathbf{H}}\leqslant 3\max\left(\frac{2r}{r^*},\max_{1\leqslant k\leqslant I(r^*)}\left|\cos(\sqrt{\lambda_{i(r^*)+k}}\,t)-\cos(\sqrt{\lambda}t)\right|\right), \tag{2.15}$$

which are bounds depending on the relation between λ, r, r^* *and* t; *here* $C_{\mathbf{V}}$, C^* *are constants independent of these parameters.*

In the case where u *belongs to* $\mathbf{V}$, *the relation* (2.2) *holds in the norm of* $\mathbf{V}$ *and* u^* *is bounded in the norm of* $\mathbf{V}$ *by a constant* C^* *independent of* r^*, *then the estimate in* (2.14) *holds for* $\|(\sqrt{\lambda})^{-1}\sin(\sqrt{\lambda}t)u-\mathbf{u}(t)\|_{\mathbf{V}}$.

Proof. The estimate (2.13) is a consequence of the conservation of the energy. Namely, consider $\varphi=0$, $\psi=u-\sum_{k=1}^{I(r^*)}\alpha_k u_{i(r^*)+k}$ in problem (2.3), which by uniqueness has the solution:

$$\mathbf{u}(t)-\sum_{k=1}^{I(r^*)}\alpha_k\frac{\sin(\sqrt{\lambda_{i(r^*)+k}}\,t)}{\sqrt{\lambda_{i(r^*)+k}}}u_{i(r^*)+k};$$

then, from (2.8) and (2.2) we easily obtain (2.13) for any t.

Let us prove (2.14). Indeed, we can write:

$$\left\|\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}u-\mathbf{u}(t)\right\|_{\mathbf{H}}$$
$$\leqslant\left\|\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}u-\sum_{k=1}^{I(r^*)}\alpha_k\frac{\sin(\sqrt{\lambda_{i(r^*)+k}}\,t)}{\sqrt{\lambda_{i(r^*)+k}}}u_{i(r^*)+k}\right\|_{\mathbf{H}}+\left\|\sum_{k=1}^{I(r^*)}\alpha_k\frac{\sin(\sqrt{\lambda_{i(r^*)+k}}\,t)}{\sqrt{\lambda_{i(r^*)+k}}}u_{i(r^*)+k}-\mathbf{u}(t)\right\|_{\mathbf{H}}.$$

On account of the continuous imbedding of $\mathbf{V}$ in $\mathbf{H}$, and of (2.13), the last term on the right-hand side of the above inequality is bounded by $(2rC_{\mathbf{V}})/r^*$. Let us consider the first term of the inequality, we can write:

$$\begin{aligned}&\left\|\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}u-\sum_{k=1}^{I(r^*)}\alpha_k\frac{\sin(\sqrt{\lambda_{i(r^*)+k}}t)}{\sqrt{\lambda_{i(r^*)+k}}}u_{i(r^*)+k}\right\|_{\mathbf{H}}\\
&\quad\leqslant\left\|\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}u-\sum_{k=1}^{I(r^*)}\alpha_k\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}u_{i(r^*)+k}\right\|_{\mathbf{H}}\\
&\qquad+\left\|\sum_{k=1}^{I(r^*)}\alpha_k\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}u_{i(r^*)+k}-\sum_{k=1}^{I(r^*)}\alpha_k\frac{\sin(\sqrt{\lambda_{i(r^*)+k}}t)}{\sqrt{\lambda_{i(r^*)+k}}}u_{i(r^*)+k}\right\|_{\mathbf{H}}.\end{aligned}$$

Again, because of (2.2), the first term on the right-hand side of the above inequality is bounded by $\sqrt{\lambda^{-1}}(2r/r^*)$. As for the second term, we take into account that λ and $\lambda_{i(r^*)+k}$ are positive and satisfy $|\lambda_{i(r^*)+k}^{-1}-\lambda^{-1}|\leqslant r^*$, for $k=1,2,\dots,I(r^*)$, then we have:

$$\begin{aligned}&\left\|\sum_{k=1}^{I(r^*)}\alpha_k\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}u_{i(r^*)+k}-\sum_{k=1}^{I(r^*)}\alpha_k\frac{\sin(\sqrt{\lambda_{i(r^*)+k}}t)}{\sqrt{\lambda_{i(r^*)+k}}}u_{i(r^*)+k}\right\|_{\mathbf{H}}\\
&\quad\leqslant\max_{1\leqslant k\leqslant I(r^*)}\left|\frac{\sin(\sqrt{\lambda_{i(r^*)+k}}t)}{\sqrt{\lambda_{i(r^*)+k}}}-\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}\right|\left\|\sum_{k=1}^{I(r^*)}\alpha_k u_{i(r^*)+k}\right\|_{\mathbf{H}}\\
&\quad=\max_{1\leqslant k\leqslant I(r^*)}\left|\frac{\sin(\sqrt{\lambda_{i(r^*)+k}}t)}{\sqrt{\lambda_{i(r^*)+k}}}-\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}\right|\\
&\quad\leqslant\max_{1\leqslant k\leqslant I(r^*)}\left|\frac{\sin(\sqrt{\lambda_{i(r^*)+k}}t)}{\sqrt{\lambda_{i(r^*)+k}}}-\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda_{i(r^*)+k}}}\right|+\max_{1\leqslant k\leqslant I(r^*)}\left|\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda_{i(r^*)+k}}}-\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}\right|\\
&\quad\leqslant\max_{1\leqslant k\leqslant I(r^*)}\frac{1}{\sqrt{\lambda_{i(r^*)+k}}}\left|\sin(\sqrt{\lambda_{i(r^*)+k}}t)-\sin(\sqrt{\lambda}t)\right|+\sqrt{r^*}\\
&\quad\leqslant\left(\frac{1}{\sqrt{\lambda}}+\sqrt{r^*}\right)\max_{1\leqslant k\leqslant I(r^*)}\left|\sin(\sqrt{\lambda_{i(r^*)+k}}t)-\sin(\sqrt{\lambda}t)\right|+\sqrt{r^*}.\end{aligned}$$

Thus, we have proved (2.14) for $C^*=1$.

Now, we follow the same steps to derive the inequality (2.15):

$$\begin{aligned}&\left\|\cos(\sqrt{\lambda}t)u-\frac{d\mathbf{u}}{dt}(t)\right\|_{\mathbf{H}}\\
&\quad\leqslant\left\|\cos(\sqrt{\lambda}t)u-\sum_{k=1}^{I(r^*)}\alpha_k\cos(\sqrt{\lambda_{i(r^*)+k}}t)u_{i(r^*)+k}\right\|_{\mathbf{H}}+\left\|\sum_{k=1}^{I(r^*)}\alpha_k\cos(\sqrt{\lambda_{i(r^*)+k}}t)u_{i(r^*)+k}-\frac{d\mathbf{u}}{dt}(t)\right\|_{\mathbf{H}}\\
&\quad\leqslant\left\|\cos(\sqrt{\lambda}t)u-\sum_{k=1}^{I(r^*)}\alpha_k\cos(\sqrt{\lambda}t)u_{i(r^*)+k}\right\|_{\mathbf{H}}+\left\|\sum_{k=1}^{I(r^*)}\alpha_k\big(\cos(\sqrt{\lambda}t)-\cos(\sqrt{\lambda_{i(r^*)+k}}t)\big)u_{i(r^*)+k}\right\|_{\mathbf{H}}+\frac{2r}{r^*},\end{aligned}$$

and we obtain (2.15).

The last assertion in the statement of the theorem is obtained rewriting the proof of (2.14) for the norm of $\mathbf{V}$ on account of

$$\left\|\sum_{k=1}^{I(r^*)}\alpha_k u_{i(r^*)+k}\right\|_{\mathbf{V}}\leqslant C^*.$$

Therefore, the theorem holds. □

Remark 2.1. It should be noted that some different bounds could be replaced by $|(\sqrt{\lambda_{i(r^*)+k}})^{-1}-(\sqrt{\lambda})^{-1}|\leqslant\sqrt{r^*}$ used throughout the proof of Theorem 2.2, but this might not be convenient when the values λ and r^* depend on a small parameter as happens in the following sections.

Also, we note that in the case where r^*, $2r/r^*$, and $|\sin(\sqrt{\lambda_{i(r^*)+k}}t)-\sin(\sqrt{\lambda}t)|$ and $|\sin(\sqrt{\lambda_{i(r^*)+k}}t)-\sin(\sqrt{\lambda}t)|$ "are small", for $k=1,2,\dots,I(r^*)$, we can assert that the bounds in the right-hand side of (2.14) and (2.15) "are also small". This means that *t can be large in the case where $\lambda_{i(r^*)+k}$, for $k=1,2,\dots,I(r^*)$, are sufficiently near of λ.* As a matter of fact, for any positive t, we have the bounds given by the following corollary.

Corollary 2.1. *Let us consider the hypothesis and notations of Theorem* 2.2*. Then, for any $t>0$, from* (2.14) *and* (2.15) *we have*:

$$\left\|\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}u-\mathbf{u}(t)\right\|_{\mathbf{H}}\leqslant 3\max\left(\frac{2rC_{\mathbf{V}}}{r^*},\frac{2r}{r^*}\frac{1}{\sqrt{\lambda}},\widetilde{C}\left(\frac{1}{\sqrt{\lambda}}+\sqrt{r^*}\right)\left(\frac{1}{\sqrt{\lambda}}-\sqrt{r^*}\right)^{-1}\sqrt{\lambda}\sqrt{r^*}t+\sqrt{r^*}\right),\tag{2.16}$$

and

$$\left\|\cos(\sqrt{\lambda}t)u-\frac{d\mathbf{u}}{dt}(t)\right\|_{\mathbf{H}}\leqslant 3\max\left(\frac{2r}{r^*},\widetilde{C}\sqrt{r^*}\left(\frac{1}{\sqrt{\lambda}}-\sqrt{r^*}\right)^{-1}\sqrt{\lambda}t\right),\tag{2.17}$$

respectively, where $\widetilde{C}$ is a constant independent of t, r and r^*.

Proof. Estimates (2.16) and (2.17) are a consequence of (2.14) and (2.15) and the Taylor series error of the sinus and cosinus functions in a neighborhood of $\sqrt{\lambda}t$. Indeed, considering (2.14), we can write:

$$\begin{aligned}&\left\|\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}u-\mathbf{u}(t)\right\|_{\mathbf{H}}\\&\quad\leqslant 3\max\left(\frac{2rC_{\mathbf{V}}}{r^*},\frac{2r}{r^*}\frac{1}{\sqrt{\lambda}},\left(\frac{1}{\sqrt{\lambda}}+\sqrt{r^*}\right)\widetilde{C}\max_{1\leqslant k\leqslant I(r^*)}\left|\sqrt{\lambda_{i(r^*)+k}}-\sqrt{\lambda}\right|t+\sqrt{r^*}\right)\\&\quad\leqslant 3\max\left(\frac{2rC_{\mathbf{V}}}{r^*},\frac{2r}{r^*}\frac{1}{\sqrt{\lambda}},\left(\frac{1}{\sqrt{\lambda}}+\sqrt{r^*}\right)\widetilde{C}\sqrt{r^*}\left(\left(\frac{1}{\sqrt{\lambda}}-\sqrt{r^*}\right)^{-1}\sqrt{\lambda}t+\sqrt{r^*}\right)\right),\end{aligned}$$

which gives (2.16), and similarly, from (2.15) we obtain (2.17). □

Remark 2.2. Let us note that only bounds (2.15) and (2.17) are independent of the constant $C_{\mathbf{V}}$ related with the continuous imbedding of $\mathbf{V}$ in $\mathbf{H}$. This is also the case where $u\in\mathbf{V}$ and the last hypothesis of Theorem 2.2 hold; we have

$$\begin{aligned}&\left\|\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}u-\mathbf{u}(t)\right\|_{\mathbf{V}}\\&\quad\leqslant 3\max\left(\frac{2r}{r^*},\frac{2r}{r^*}\frac{1}{\sqrt{\lambda}},C^*\widetilde{C}\left(\frac{1}{\sqrt{\lambda}}+\sqrt{r^*}\right)\left(\frac{1}{\sqrt{\lambda}}-\sqrt{r^*}\right)^{-1}\sqrt{\lambda}\sqrt{r^*}t+C^*\sqrt{r^*}\right).\end{aligned}\tag{2.18}$$

The constant $C_{\mathbf{V}}$ would appear accompanying the term $2r/r^*$ on the right-hand side of (2.13) in the case where, when applying Theorem 2.1, properties (2.1) and (2.2) hold in the norm of $\mathbf{V}$ instead of $\mathbf{H}$. Consequently, this constant $C_{\mathbf{V}}$ would also affect (2.15) and (2.17). As noticed in Remark 2.1, taking into account this constant can be important for specific bounds when it also depends on a small parameter of perturbation ε.

Under the last hypothesis in Theorem 2.2, similar bounds to those in Theorem 2.2 can be obtained for the initial data $\varphi=u$, $\psi=0$, considering $\cos(\sqrt{\lambda}t)u$ instead of $(\sqrt{\lambda})^{-1}\sin(\sqrt{\lambda}t)u$. As a matter of fact, the estimates in the following theorem hold.

Theorem 2.3. *Let (u,λ^{-1}) be a quasimode with remainder r of the operator $\mathcal{A}=A_H^{-1}$, A arising in* (2.3)*. Let $\{\lambda^{-1}_{i(r^*)+k}\}_{k=1,2,\dots,I(r^*)}$ and $\{u_{i(r^*)+k}\}_{k=1,2,\dots,I(r^*)}$ be the eigenvalues and the associated eigenfunctions of the operator $\mathcal{A}$ satisfying* (2.1)–(2.2)*. Let us assume that $r^*>r$ and $\lambda^{-1}-r^*>0$. Moreover, let us assume that u belongs to $\mathbf{V}$, the relation* (2.2) *is verified in the norm of $\mathbf{V}$ and u^* is bounded in the norm of $\mathbf{V}$ by a constant C^* independent of r^*. Then, for $\varphi=u$, $\psi=0$, the solution $\mathbf{u}(t)$ of* (2.3) *satisfies*:

$$a\left(\mathbf{u}(t)-\sum_{k=1}^{I(r^*)}\alpha_k\cos\left(\sqrt{\lambda_{i(r^*)+k}}t\right)u_{i(r^*)+k},\mathbf{u}(t)-\sum_{k=1}^{I(r^*)}\alpha_k\cos\left(\sqrt{\lambda_{i(r^*)+k}}t\right)u_{i(r^*)+k}\right)$$
$$+\left\|\frac{d\mathbf{u}}{dt}(t)+\sum_{k=1}^{I(r^*)}\alpha_k\sqrt{\lambda_{i(r^*)+k}}\sin\left(\sqrt{\lambda_{i(r^*)+k}}t\right)u_{i(r^*)+k}\right\|_{\mathbf{H}}^2\leqslant\left(\frac{2r}{r^*}\right)^2,\quad\forall t>0. \tag{2.19}$$

In addition, for any $t>0$, we have:

$$\left\|\cos(\sqrt{\lambda}t)u-\mathbf{u}(t)\right\|_{\mathbf{V}}\leqslant 3\max\left(\frac{2r}{r^*},\max_{1\leqslant k\leqslant I(r^*)}\left|\cos\left(\sqrt{\lambda_{i(r^*)+k}}t\right)-\cos(\sqrt{\lambda}t)\right|\right), \tag{2.20}$$

and

$$\left\|\sqrt{\lambda}\sin(\sqrt{\lambda}t)u+\frac{d\mathbf{u}}{dt}(t)\right\|_{\mathbf{H}}$$
$$\leqslant 3\max\left(\frac{2rC}{r^*},\frac{2r}{r^*}\sqrt{\lambda},C^*\left(\frac{1}{\sqrt{\lambda}}-\sqrt{r^*}\right)^{-1}\left(\max_{1\leqslant k\leqslant I(r^*)}\left|\sin\left(\sqrt{\lambda_{i(r^*)+k}}t\right)-\sin(\sqrt{\lambda}t)\right|+\sqrt{r^*}\sqrt{\lambda}\right)\right), \tag{2.21}$$

which are bounds depending on the relation between λ, r, r^* and t; C and C^* are constants independent of these parameters.

Now, following the idea of Corollary 2.1, we derive bounds for,

$$\left\|\cos(\sqrt{\lambda}t)u-\mathbf{u}(t)\right\|_{\mathbf{V}}\quad\text{and}\quad\left\|\sqrt{\lambda}\sin(\sqrt{\lambda}t)u+\frac{d\mathbf{u}}{dt}(t)\right\|_{\mathbf{H}},$$

in terms of t, λ, r^* and r.

Corollary 2.2. *Let us consider the hypothesis and notations of Theorem* 2.3. *Then, for any $t>0$, from* (2.20) *and* (2.21) *we have*:

$$\left\|\cos(\sqrt{\lambda}t)u-\mathbf{u}(t)\right\|_{\mathbf{V}}\leqslant 3\max\left(\frac{2r}{r^*},\widetilde{C}\sqrt{r^*}\left(\frac{1}{\sqrt{\lambda}}-\sqrt{r^*}\right)^{-1}\sqrt{\lambda}t\right), \tag{2.22}$$

and

$$\left\|\sqrt{\lambda}\sin(\sqrt{\lambda}t)u+\frac{d\mathbf{u}}{dt}(t)\right\|_{\mathbf{H}}$$
$$\leqslant 3\max\left(\frac{2rC}{r^*},\frac{2r}{r^*}\sqrt{\lambda},\widetilde{C}\left(\frac{1}{\sqrt{\lambda}}-\sqrt{r^*}\right)^{-1}\left(\left(\frac{1}{\sqrt{\lambda}}-\sqrt{r^*}\right)^{-1}\sqrt{\lambda}\sqrt{r^*}t+\sqrt{r^*}\sqrt{\lambda}\right)\right), \tag{2.23}$$

respectively, where C and $\widetilde{C}$ are constants independent of t, r and r^*.

Remark 2.3. We emphasize that the bounds (2.14)–(2.17) ((2.20)– (2.23), respectively) establish the range of t where the *standing wave* $(\sqrt{\lambda})^{-1}\sin(\sqrt{\lambda}t)u$ ($\cos(\sqrt{\lambda}t)u$, respectively) approaches the solution $\mathbf{u}(t)$ of problem (2.3) for the initial data $\varphi=0$ and $\psi=u$ ($\varphi=u$ and $\psi=0$, respectively), (u,λ^{-1}) being a given quasimode of the operator $\mathcal{A}$. It should also be mentioned that, here and hereafter, the so-called standing waves do not satisfy in general homogeneous wave equations.

Remark 2.4. Under the hypothesis in Theorem 2.3, given the initial data $\varphi=u$, $\psi=\sqrt{\lambda}u$, rewriting proofs in Theorems 2.2 and 2.3 leads us to obtain approaches for solutions of (2.3) via the function $\cos(\sqrt{\lambda}t)u+\sin(\sqrt{\lambda}t)u$ (see (2.11) and (2.12)).

Also, extensions of the results in Theorems 2.2 and 2.3 and Corollaries 2.1 and 2.2, for more general initial data can be obtained combining the results in both theorems (see (2.9) and (2.10)): namely, for $\varphi=u$ and $\psi=v$, where (u,λ_1^{-1}) and (v,λ_2^{-1}) are quasimodes of the operator $\mathcal{A}$, approaches to solutions of (2.3) are given by sums of standing waves of different frequencies, namely $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$.

Remark 2.5. It is self-evident that all the results in Theorems 2.2 and 2.3 and Corollaries 2.1 and 2.2 hold in the case where we know that there is only one eigenvalue $\lambda_{i(r^*)}^{-1}$ in the interval $[\lambda^{-1}-r^*, \lambda^{-1}+r^*]$. In this case, the restriction $\lambda^{-1}-r^*>a>0$ allows us to assert that $\|u^*\|_{\mathbf{V}}$ is bounded independently of r^*.

Remark 2.6. Note that proofs of the results throughout the section rely on the fact that properties (2.1) and (2.2) hold in spaces **V** and/or **H** more than in the precise definition of quasimode at the beginning of the section.

3. The spectral perturbation problem

It appears in the literature of the spectral perturbation theory that when applying Theorem 2.1, the spaces and operators under consideration often depend on a small parameter ε that converges towards 0. Also, the functions u and numbers λ and r arising in the definition of a quasimode depend on this parameter. We establish here an abstract general framework that can be applied to several problems of spectral perturbation theory.

Let us consider ε a small parameter $\varepsilon\in(0,1)$. Let $\mathcal{V}^\varepsilon$ and $\mathcal{H}^\varepsilon$ be two separable Hilbert spaces and $\mathcal{V}^\varepsilon\subset\mathcal{H}^\varepsilon$, with dense and compact imbedding. Let $a^\varepsilon(u,v)$ be a sesquilinear, hermitian, continuous and coercive form on $\mathcal{V}^\varepsilon$. We consider $\mathcal{V}^\varepsilon$ equipped with the scalar product inducted by $a^\varepsilon(\cdot,\cdot)$, namely $\langle u,v\rangle_{\mathcal{V}^\varepsilon}=a^\varepsilon(u,v)$. Let $A^\varepsilon\in\mathcal{L}(\mathcal{V}^\varepsilon,(\mathcal{V}^\varepsilon)')$ be the operator associated with the form a^ε, namely, $a^\varepsilon(u,v)=\langle A^\varepsilon u,v\rangle_{(\mathcal{V}^\varepsilon)'\times\mathcal{V}^\varepsilon}$. Let us assume that

$$\|u\|_{\mathcal{H}^\varepsilon}\leqslant C\|u\|_{\mathcal{V}^\varepsilon},\quad \forall u\in\mathcal{V}^\varepsilon, \tag{3.1}$$

where C is a constant independent of u and ε.

Let $A^\varepsilon_{\mathcal{H}^\varepsilon}$ be the operator restriction of A^ε to $\mathcal{H}^\varepsilon$, with domain of definition $D(A^\varepsilon_{\mathcal{H}^\varepsilon})=\{v\in\mathcal{V}^\varepsilon/A^\varepsilon v\in\mathcal{H}^\varepsilon\}$. Then, $\mathcal{A}^\varepsilon=(A^\varepsilon_{\mathcal{H}^\varepsilon})^{-1}$, $\mathcal{A}^\varepsilon:\mathcal{H}^\varepsilon\to\mathcal{H}^\varepsilon$ is an operator which satisfies the properties in Theorem 2.1. The eigenvalues of A^ε (respectively, $\mathcal{A}^\varepsilon$) are $\{\lambda_i^\varepsilon\}_{i=1}^\infty$ (respectively $\{(\lambda_i^\varepsilon)^{-1}\}_{i=1}^\infty$), and the associated eigenfunctions are $\{u_i^\varepsilon\}_{i=1}^\infty$ which form an orthogonal basis in $\mathcal{H}^\varepsilon$ and $\mathcal{V}^\varepsilon$, u_i^ε of norm 1 in $\mathcal{H}^\varepsilon$ and of norm $\sqrt{\lambda_i^\varepsilon}$ in $\mathcal{V}^\varepsilon$.

Let us consider the evolution problem:

$$\begin{cases}\dfrac{d^2\mathbf{u}^\varepsilon}{dt^2}+A^\varepsilon\mathbf{u}^\varepsilon=0,\\ \mathbf{u}^\varepsilon(0)=\varphi^\varepsilon,\\ \dfrac{d\mathbf{u}^\varepsilon}{dt}(0)=\psi^\varepsilon.\end{cases} \tag{3.2}$$

For initial data $(\varphi^\varepsilon,\psi^\varepsilon)\in\mathcal{V}^\varepsilon\times\mathcal{H}^\varepsilon$, there is a unique solution $\mathbf{u}^\varepsilon(t)$ of (3.2) which satisfies properties (2.4)–(2.8) for $(\varphi,\psi)\equiv(\varphi^\varepsilon,\psi^\varepsilon)$, $\mathbf{V}\equiv\mathcal{V}^\varepsilon$ and $\mathbf{H}\equiv\mathcal{H}^\varepsilon$.

Theorem 3.1. *For each fixed ε, let $(u^\varepsilon,\lambda^{-1})$ be a quasimode with remainder r^ε of the operator $\mathcal{A}^\varepsilon=(A^\varepsilon_{\mathcal{H}^\varepsilon})^{-1}$, A^ε arising in (3.2). Let r^*_ε be $r^*_\varepsilon>r^\varepsilon$, $r^\varepsilon\to0$ and $r^*_\varepsilon\to0$ as $\varepsilon\to0$. On account of Theorem 2.1, let $\{(\lambda^\varepsilon_{i(r^*)+k})^{-1}\}_{k=1,2,\dots,I(r^*_\varepsilon)}$ be the eigenvalues of $\mathcal{A}^\varepsilon$ in the interval $[\lambda^{-1}-r^*_\varepsilon,\lambda^{-1}+r^*_\varepsilon]$, for some index $i(r^*_\varepsilon)$ and natural number $I(r^*_\varepsilon)\geqslant1$, and let $\{u^\varepsilon_{i(r^*)+k}\}_{k=1,2,\dots,I(r^*_\varepsilon)}$ be the associated eigenfunctions. Let $u^{\varepsilon*}\in\mathcal{H}^\varepsilon$,*

$$\|u^{\varepsilon*}\|_{\mathcal{H}^\varepsilon}=1,\quad u^{\varepsilon*}\in\big[u^\varepsilon_{i(r^*_\varepsilon)+1},u^\varepsilon_{i(r^*_\varepsilon)+2},\dots,u^\varepsilon_{i(r^*_\varepsilon)+I(r^*_\varepsilon)}\big],\qquad u^{\varepsilon*}=\sum_{k=1}^{I(r^*_\varepsilon)}\alpha^\varepsilon_k u^\varepsilon_{i(r^*_\varepsilon)+k}, \tag{3.3}$$

satisfying

$$\|u^\varepsilon-u^{\varepsilon*}\|_{\mathcal{H}^\varepsilon}=\Bigg\|u^\varepsilon-\sum_{k=1}^{I(r^*_\varepsilon)}\alpha^\varepsilon_k u^\varepsilon_{i(r^*_\varepsilon)+k}\Bigg\|_{\mathcal{H}^\varepsilon}\leqslant\frac{2r^\varepsilon}{r^*_\varepsilon}, \tag{3.4}$$

where the α^ε_k are constants such that $|\alpha^\varepsilon_k|\leqslant1$ for $k=1,2,\dots,I(r^*_\varepsilon)$. In addition, let us assume that $\lim_{\varepsilon\to0}(r^\varepsilon/r^*_\varepsilon)=0$ and that there is a constant $\delta>0$ such that $\lambda^{-1}-r^*_\varepsilon>\delta$ for sufficiently small ε.

Then, for $\varphi^\varepsilon = 0$, $\psi^\varepsilon = u^\varepsilon$*, the solution* $\mathbf{u}^\varepsilon(t)$ *of* (3.2) *satisfies*:

$$a^\varepsilon\left(\mathbf{u}^\varepsilon(t) - \sum_{k=1}^{I(r_\varepsilon^*)} \alpha_k^\varepsilon \frac{\sin(\sqrt{\lambda^\varepsilon_{i(r^*)+k}}t)}{\sqrt{\lambda^\varepsilon_{i(r^*)+k}}} u^\varepsilon_{i(r^*)+k}, \mathbf{u}^\varepsilon(t) - \sum_{k=1}^{I(r_\varepsilon^*)} \alpha_k^\varepsilon \frac{\sin(\sqrt{\lambda^\varepsilon_{i(r^*)+k}}t)}{\sqrt{\lambda^\varepsilon_{i(r^*)+k}}} u^\varepsilon_{i(r^*)+k}\right)$$
$$+ \left\| \frac{d\mathbf{u}^\varepsilon}{dt}(t) - \sum_{k=1}^{I(r_\varepsilon^*)} \alpha_k^\varepsilon \cos\left(\sqrt{\lambda_{i(r_\varepsilon^*)+k}}t\right) u_{i(r_\varepsilon^*)+k}\right\|^2_{\mathcal{H}^\varepsilon} \leqslant \left(\frac{2r^\varepsilon}{r_\varepsilon^*}\right)^2, \quad \forall t > 0. \tag{3.5}$$

In addition, for any $t > 0$ *and sufficiently small* ε*, namely* $\varepsilon < \varepsilon_0$ *with* ε_0 *independent of* t*, the estimates*:

$$\left\| \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} u^\varepsilon - \mathbf{u}^\varepsilon(t)\right\|_{\mathcal{H}^\varepsilon} \leqslant \max\left(C_1 \frac{r^\varepsilon}{r_\varepsilon^*}, C_2 \frac{r^\varepsilon}{r_\varepsilon^*}, C_3\sqrt{r_\varepsilon^*}t + C_4\sqrt{r_\varepsilon^*}\right), \tag{3.6}$$

and

$$\left\| \cos(\sqrt{\lambda}t)u^\varepsilon - \frac{d\mathbf{u}^\varepsilon}{dt}(t)\right\|_{\mathcal{H}^\varepsilon} \leqslant \max\left(\frac{2r_\varepsilon}{r_\varepsilon^*}, C_5\sqrt{r_\varepsilon^*}t\right), \tag{3.7}$$

hold, where C_1, C_2, C_3, C_4 *and* C_5 *are constants that may depend on* λ *but which are independent of* t *and* ε.

In the case where u^ε *belongs to* $\mathcal{V}^\varepsilon$*, the relation* (3.4) *is satisfied for the norm of* $\mathcal{V}^\varepsilon$*, and* $u^{\varepsilon *}$ *is bounded in the norm of* $\mathcal{V}^\varepsilon$ *by a constant* C^* *independent of* ε*, that is the inequality,*

$$\|u^\varepsilon - u^{\varepsilon *}\|_{\mathcal{V}^\varepsilon} = \left\| u^\varepsilon - \sum_{k=1}^{I(r_\varepsilon^*)} \alpha_k^\varepsilon u^\varepsilon_{i(r_\varepsilon^*)+k}\right\|_{\mathcal{V}^\varepsilon} \leqslant \widetilde{C}\frac{2r^\varepsilon}{r_\varepsilon^*}, \quad \text{and} \quad \left\| \sum_{k=1}^{I(r_\varepsilon^*)} \alpha_k^\varepsilon u^\varepsilon_{i(r_\varepsilon^*)+k}\right\|_{\mathcal{V}^\varepsilon} \leqslant \widetilde{C}^*, \tag{3.8}$$

holds for sufficiently small ε *and for certain constants* $\widetilde{C}$ *and* $\widetilde{C}^*$ *independent of* ε*, then we also have*

$$\left\| \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} u^\varepsilon - \mathbf{u}^\varepsilon(t)\right\|_{\mathcal{V}^\varepsilon} \leqslant \max\left(C_1 \frac{r^\varepsilon}{r_\varepsilon^*}, C_2 \frac{r^\varepsilon}{r_\varepsilon^*}, C_3\sqrt{r_\varepsilon^*}t + C_4\sqrt{r_\varepsilon^*}\right). \tag{3.9}$$

Theorem 3.2. *Let us consider all the hypothesis of Theorem* 3.1 *for* λ, r^ε, r_ε^*, $u^\varepsilon \in \mathcal{V}^\varepsilon$ *and* $u^{\varepsilon *}$ *satisfying* (3.3)–(3.4) *and* (3.8)*. Then, for* $\varphi = u^\varepsilon$, $\psi^\varepsilon = 0$*, the unique solution* $\mathbf{u}^\varepsilon(t)$ *of* (3.2) *satisfies*:

$$a^\varepsilon\left(\mathbf{u}^\varepsilon(t) - \sum_{k=1}^{I(r_\varepsilon^*)} \alpha_k^\varepsilon \cos\left(\sqrt{\lambda^\varepsilon_{i(r_\varepsilon^*)+k}}t\right) u^\varepsilon_{i(r^*)+k}, \mathbf{u}^\varepsilon(t) - \sum_{k=1}^{I(r_\varepsilon^*)} \alpha_k^\varepsilon \cos\left(\sqrt{\lambda^\varepsilon_{i(r_\varepsilon^*)+k}}t\right) u_{i(r_\varepsilon^*)+k}\right)$$
$$+ \left\| \frac{d\mathbf{u}^\varepsilon}{dt}(t) + \sum_{k=1}^{I(r_\varepsilon^*)} \alpha_k \sqrt{\lambda^\varepsilon_{i(r_\varepsilon^*)+k}} \sin\left(\sqrt{\lambda^\varepsilon_{i(r_\varepsilon^*)+k}}t\right) u_{i(r_\varepsilon^*)+k}\right\|^2_{\mathcal{H}^\varepsilon} \leqslant \left(\frac{2r^\varepsilon}{r_\varepsilon^*}\right)^2, \quad \forall t > 0. \tag{3.10}$$

In addition, for $t > 0$*, and sufficiently small* ε*, namely* $\varepsilon < \varepsilon_0$ *with* ε_0 *independent of* t*, the estimates*:

$$\left\| \cos(\sqrt{\lambda}t)u^\varepsilon - \mathbf{u}^\varepsilon(t)\right\|_{\mathcal{V}^\varepsilon} \leqslant \max\left(C_1 \frac{r^\varepsilon}{r_\varepsilon^*}, C_2\sqrt{r_\varepsilon^*}t\right), \tag{3.11}$$

and

$$\left\| \sqrt{\lambda}\sin(\sqrt{\lambda}t)u^\varepsilon + \frac{d\mathbf{u}^\varepsilon}{dt}(t)\right\|_{\mathcal{H}^\varepsilon} \leqslant \max\left(C_1 \frac{r^\varepsilon}{r_\varepsilon^*}, C_3 \frac{r^\varepsilon}{r_\varepsilon^*}, C_4\sqrt{r_\varepsilon^*}t + C_5\sqrt{r_\varepsilon^*}\right), \tag{3.12}$$

hold, where C_1, C_2, C_3, C_4 *and* C_5 *are constants that may depend on* λ *but which are independent of* t *and* ε.

On account of (3.1), the proofs of Theorems 3.1 and 3.2 follow applying the results in Theorem 2.2 and Corollary 2.1 and Theorem 2.3 and Corollary 2.2 respectively with minor modifications: specifying, the appropriate modifications of spaces, functions and constants which now depend on ε.

Remark 3.1. Under the assumptions $\|u^\varepsilon\|_{\mathcal{V}^\varepsilon} = 1$, $\|u^{\varepsilon *}\|_{\mathcal{V}^\varepsilon} = 1$, (3.1) and (3.8), similar relations to those in (3.3), (3.4) hold, namely,

$$\|u^\varepsilon\|_{\mathcal{H}^\varepsilon} \leqslant C \quad \text{and} \quad \|u^\varepsilon - u^{\varepsilon *}\|_{\mathcal{H}^\varepsilon} \leqslant C\frac{r^\varepsilon}{r^*_\varepsilon},$$

for C a constant independent of ε, and the estimates in Theorems 3.1 and 3.2 also hold.

Let us note that property (3.1) always holds in the case where $\mathcal{V}^\varepsilon$ is equipped with the norm $\|u\|_{\mathcal{H}^\varepsilon} + \|u\|_{\mathcal{V}^\varepsilon}$ which is equivalent to the norm of $\mathcal{V}^\varepsilon$ given by $a^\varepsilon(u,u)^{1/2}$ (cf. Remark 2.2 to compare). With this new norm for $\mathcal{V}^\varepsilon$, bounds (3.8) imply the kind of bounds in (3.3)–(3.4).

Remark 3.2. Note that the bounds in Theorems 3.1 and 3.2 hold on the basis that λ is independent of ε. In contrast, the bounds in Theorems 2.2 and 2.3 and Corollaries 2.1 and 2.2 make it to maintain possible the dependence on ε of all the constants λ, r and r^*. Also we note that Remarks 2.4 and 2.5, with the suitable modifications, apply to the results in this section where spaces, operators, quasimodes and rests depend on ε.

Remark 3.3. It should be noted that estimates (3.6), (3.7), (3.9), (3.11) and (3.12) establish the range of t for which the standing waves $\cos(\sqrt{\lambda}t)u^\varepsilon$ or $\sin(\sqrt{\lambda}t)u^\varepsilon$ approach the solutions of (3.2), when the given initial data (along with λ) are quasimodes of $\mathcal{A}^\varepsilon$. This range of t is $t \in [0, (r^*_\varepsilon)^{-\alpha/2}]$ for any constant α with $0 < \alpha < 1$, and the precise bounds for the discrepancies between the solutions and the standing waves depend on the relations between $r^\varepsilon/r^*_\varepsilon$, $\sqrt{r^*_\varepsilon}$ and $\sqrt{r^*_\varepsilon}t$.

In connection with Remark 3.1, we note that the hypothesis of the uniform continuous imbedding (3.1) of $\mathcal{V}^\varepsilon$ into $\mathcal{H}^\varepsilon$ can be weakened but this also implies weakening the estimates (3.6), (3.7), (3.9), (3.11) and (3.12) involving norms in $\mathcal{H}^\varepsilon$, and more specifically those related to the derivatives with respect to time. As a matter of fact, when applying the general results of Theorems 3.1 and 3.2 to particular vibrating systems, it may occur that the norm of the space $\mathcal{H}^\varepsilon$ involves a singular weight $\rho^\varepsilon(x)$ depending on ε and neither (3.1) nor (3.3)–(3.4) hold, while it is possible to show estimates (3.8) for $\|u^\varepsilon\|_{\mathcal{V}^\varepsilon} = 1$ (cf., for instance, Theorem 4.6 and Remark 4.8). Roughly speaking, this amounts to saying that we can apply Theorem 2.1 when the space $\mathbf{H}$ is $\mathcal{V}^\varepsilon$ instead of $\mathcal{H}^\varepsilon$. In this case, we can only prove the estimate (3.11) as stated in the following theorem.

Theorem 3.3. *For each fixed ε, let $u^\varepsilon \in \mathcal{V}^\varepsilon$, with $\|u^\varepsilon\|_{\mathcal{V}^\varepsilon} = 1$, and let λ^{-1} be a positive number, and r_ε and r^*_ε be such that $r^*_\varepsilon > r^\varepsilon$, $r^\varepsilon \to 0$ and $r^*_\varepsilon \to 0$ as $\varepsilon \to 0$. Let us assume that $\{(\lambda^\varepsilon_{i(r^*)+k})^{-1}\}_{k=1,2,\dots,I(r^*_\varepsilon)}$ are eigenvalues of the operator A^ε arising in (3.2) which are in the interval $[\lambda^{-1} - r^*_\varepsilon, \lambda^{-1} + r^*_\varepsilon]$, for some index $i(r^*_\varepsilon)$ and natural number $I(r^*_\varepsilon) \geqslant 1$, and let $\{u^\varepsilon_{i(r^*)+k}\}_{k=1,2,\dots,I(r^*_\varepsilon)}$ be the associated eigenfunctions. Let us assume that there is $u^{\varepsilon *} \in \mathcal{V}^\varepsilon$,*

$$\|u^{\varepsilon *}\|_{\mathcal{V}^\varepsilon} = 1, \quad u^{\varepsilon *} \in \left[u^\varepsilon_{i(r^*_\varepsilon)+1}, u^\varepsilon_{i(r^*_\varepsilon)+2}, \dots, u^\varepsilon_{i(r^*_\varepsilon)+I(r^*_\varepsilon)}\right], \quad u^{\varepsilon *} = \sum_{k=1}^{I(r^*_\varepsilon)} \alpha^\varepsilon_k u^\varepsilon_{i(r^*_\varepsilon)+k}, \tag{3.13}$$

satisfying

$$\|u^\varepsilon - u^{\varepsilon *}\|_{\mathcal{V}^\varepsilon} = \left\|u^\varepsilon - \sum_{k=1}^{I(r^*_\varepsilon)} \alpha^\varepsilon_k u^\varepsilon_{i(r^*_\varepsilon)+k}\right\|_{\mathcal{V}^\varepsilon} \leqslant \frac{2r^\varepsilon}{r^*_\varepsilon}, \tag{3.14}$$

where, the α^ε_k are constants such that $|\alpha^\varepsilon_k| \leqslant 1$ for $k = 1, 2, \dots, I(r^*_\varepsilon)$. In addition, let us assume that $\lim_{\varepsilon\to 0}(r^\varepsilon/r^*_\varepsilon) = 0$ and that there is a constant $\delta > 0$ such that $\lambda^{-1} - r^*_\varepsilon > \delta$ for sufficiently small ε. For $\varphi^\varepsilon = u$, $\psi^\varepsilon = 0$, let $\mathbf{u}^\varepsilon(t)$ denote the solution of problem (3.2). Then, for any $t > 0$, and sufficiently small ε (namely $\varepsilon < \varepsilon_0$ with ε_0 independent of t) we have

$$\left\|\cos(\sqrt{\lambda}t)u^\varepsilon - \mathbf{u}^\varepsilon(t)\right\|_{\mathcal{V}^\varepsilon} \leqslant \max\left(C_1\frac{r^\varepsilon}{r^*_\varepsilon}, C_2\sqrt{r^*_\varepsilon}t\right), \tag{3.15}$$

where C_1 and C_2 are constants that may depend on λ but which are independent of t and ε.

4. Vibrating systems with concentrated masses

In this section, we consider vibrating systems with many concentrated masses which have been described first in [20–22]. Extensions of the results and comparisons with other vibrating systems in the literature are also outlined in Remarks 4.2–4.11.

The spectral problem here considered, problem (4.2), depends on a small parameter ε which deals with the size of the masses B^ε, their number $N(\varepsilon)$, and their densities $\rho^\varepsilon(x)$. The asymptotic behavior, as $\varepsilon \to 0$, for the eigenelements $(\lambda^\varepsilon, u^\varepsilon)$ of (4.2) has been studied in many papers: let us refer to [9] and [26] for further references.

In the case where $\lambda^\varepsilon = O(\varepsilon^{m-2})$, results on the construction of quasimodes to approach low frequency vibrations can be found in [24,32,33], while those for $\lambda^\varepsilon = O(1)$ can be found in [25]. Here, considering the second order evolution problem (4.5), we provide estimates for the time t in which certain *standing waves* approach time dependent solutions when the initial data are quasimodes. Also, precise bounds for the discrepancies between the solutions and the standing waves in the suitable Hilbert spaces are obtained. In Section 4.1, we provide a summary of the results obtained in previous papers for the low frequencies and the microscopic variable (cf. Theorem 4.1) while interpreting them in the macroscopic variable (cf. Theorem 4.2). In Section 4.2 we obtain the approaches for the solutions of the associated wave equations in both variables, the macroscopic variable being the original one. It should be noted that a re-scaling of the eigenvalues of the original ε dependent spectral problem leads to a re-scaling of times and velocities in terms of the solutions of the wave equations (see Theorems 4.3 and 4.4 and Remarks 4.1 and 4.5 to compare). Finally, Section 4.3 contains a brief summary of the results for the high frequency vibrations, which have been obtained in previous papers, and the application of these results to approach solutions of the evolution problem (4.5) via standing waves which concentrate their support over the whole domain for large times.

Let Ω be any bounded domain of $\mathbb{R}^n$, $n = 2, 3$, with a Lipschitz boundary $\partial\Omega$. Let Σ and Γ_Ω be non-empty parts of the boundary, such that $\partial\Omega = \overline{\Sigma} \cup \overline{\Gamma}_\Omega$, and Σ is assumed to be in contact with $\{x_n = 0\}$. Let ε and η be two small parameters such that $\varepsilon \ll \eta$ and $\eta = \eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

For $n = 2$, let B be the semi-circle $B = \{(y_1, y_2) \,/\, y_1^2 + y_2^2 < 1,\, y_2 < 0\}$ in the auxiliary space $\mathbb{R}^2$ with coordinates y_1, y_2. For $n = 3$, let B be the half-ball $B = \{(y_1, y_2, y_3) \,/\, y_1^2 + y_2^2 + y_3^2 < 1,\, y_3 < 0\}$ in the auxiliary space $\mathbb{R}^3$ with coordinates y_1, y_2, y_3. Let ∂B be the boundary of B, $\partial B = \overline{T} \cup \overline{\Gamma}$, where T is the part lying on $\{y_n = 0\}$. Let B^ε (and similarly T^ε, Γ^ε) denote its homothetic εB (εT, $\varepsilon\Gamma$). Let B_k^ε (and similarly T_k^ε, Γ_k^ε) denote the domain obtained by translation of the previous B^ε (T^ε, Γ^ε) centered at the point $\tilde{x}_k$ of Σ at distance η between them. k is a parameter ranging from 1 to $N(\varepsilon)$, $k \in \mathbb{N}$. $N(\varepsilon)$ denotes the number of B_k^ε contained in Ω; $N(\varepsilon)$ is of order $O(\frac{1}{\eta})$ when $n = 2$ and $O(\frac{1}{\eta^2})$ when $n = 3$. The parameter α denotes the value:

$$\alpha = \lim_{\varepsilon\to 0} \frac{-1}{\eta \ln \varepsilon} \quad \text{when } n = 2 \quad \text{and} \quad \alpha = \lim_{\varepsilon\to 0} \frac{\varepsilon}{\eta^2} \quad \text{when } n = 3. \tag{4.1}$$

We consider the eigenvalue problem:

$$\begin{cases} -\Delta u^\varepsilon = \rho^\varepsilon(x)\lambda^\varepsilon u^\varepsilon & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \Gamma_\Omega \cup \bigcup T^\varepsilon, \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \Sigma - \overline{\bigcup T^\varepsilon}, \end{cases} \tag{4.2}$$

where $\rho^\varepsilon = \rho^\varepsilon(x)$ is the density function defined as

$$\rho^\varepsilon(x) = \frac{1}{\varepsilon^m} \quad \text{if } x \in \bigcup B^\varepsilon, \qquad \rho^\varepsilon(x) = 1 \quad \text{if } x \in \Omega - \overline{\bigcup B^\varepsilon}. \tag{4.3}$$

The symbol $\bigcup$ is extended, for fixed ε, to all the regions B_k^ε contained in Ω and the parameter m is a real number, $m > 2$ (see [20–22], for different values of the parameter m, boundary conditions and shapes of the domains).

As is well known, problem (4.2) has a discrete spectrum. For fixed ε, let $\{\lambda_i^\varepsilon\}_{i=1}^\infty$ be the sequence of eigenvalues of (4.2) converging to ∞, with the classical convention of repeated eigenvalues. It has been proved (see [20–22]) that they satisfy the estimates $C\varepsilon^{m-2} \leqslant \lambda_i^\varepsilon \leqslant C_i\varepsilon^{m-2}$, where C is a constant independent of ε and i and C_i is a constant independent of ε. Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the corresponding sequence of eigenfunctions which are an orthogonal basis of the space $\mathbf{V}^\varepsilon$, where $\mathbf{V}^\varepsilon$ is the completion of $\{u \in \mathcal{D}(\overline{\Omega})/u = 0 \text{ on } \Gamma_\Omega \cup \bigcup T^\varepsilon\}$ in the topology of $H^1(\Omega)$. The norm that we consider in $\mathbf{V}^\varepsilon$ is that of the gradient $\|\nabla_x u\|_{L^2(\Omega)}$. In fact, the variational formulation of the problem reads:

$$\int_\Omega \nabla u^\varepsilon . \nabla v^\varepsilon \, dx = \lambda^\varepsilon \int_\Omega \rho^\varepsilon(x) u^\varepsilon v^\varepsilon \, dx, \quad \forall v^\varepsilon \in \mathbf{V}^\varepsilon. \tag{4.4}$$

Let us consider the set of functional spaces $\mathbf{V}^\varepsilon$ and $\mathbf{H}^\varepsilon$, $\mathbf{H}^\varepsilon = L^2(\Omega)$ with the norm $\|(\rho^\varepsilon)^{1/2} u\|_{L^2(\Omega)}$, ρ^ε being defined by (4.3). Let A^ε be the operator associated with the form on $\mathbf{V}^\varepsilon$ arising on the left-hand side of (4.4). Let us consider the evolution problem associated with (4.4):

$$\begin{cases} \dfrac{d^2\mathbf{u}^\varepsilon}{dt^2} + A^\varepsilon \mathbf{u}^\varepsilon = 0, \\ \mathbf{u}^\varepsilon(0) = \varphi^\varepsilon, \\ \dfrac{d\mathbf{u}^\varepsilon}{dt}(0) = \psi^\varepsilon. \end{cases} \tag{4.5}$$

As for problem (2.3), the initial data $(\varphi^\varepsilon, \psi^\varepsilon) \in \mathbf{V}^\varepsilon \times \mathbf{H}^\varepsilon$ ensure that (4.5) has a unique solution $\mathbf{u}^\varepsilon(t)$ satisfying (2.4), (2.5) and (2.6) in the couple of spaces $(\mathbf{V}, \mathbf{H}) \equiv (\mathbf{V}^\varepsilon, \mathbf{H}^\varepsilon)$, which now reads:

$$\mathbf{u}^\varepsilon \in L^\infty\big(0, \infty, \mathbf{V}^\varepsilon\big), \qquad \frac{d\mathbf{u}^\varepsilon}{dt} \in L^\infty\big(0, \infty, \mathbf{H}^\varepsilon\big), \qquad \mathbf{u}^\varepsilon(0) = \varphi^\varepsilon,$$

and, for any fixed $T > 0$,

$$\int_0^T \left(\int_\Omega \nabla \mathbf{u}^\varepsilon . \nabla \boldsymbol{\Phi} \, dx - \int_\Omega \rho^\varepsilon(x) \frac{d\mathbf{u}^\varepsilon}{dt} \frac{d\boldsymbol{\Phi}}{dt} \, dx \right) dt = \int_\Omega \rho^\varepsilon(x) \psi^\varepsilon \boldsymbol{\Phi}(0) \, dx,$$

for any test function $\boldsymbol{\Phi}$ of the form $\boldsymbol{\Phi} = \phi(t)v$, where $v \in \mathbf{V}^\varepsilon$, and $\phi \in C^1([0,T])/\phi(T) = 0$. In addition, here (2.8) amounts to

$$\int_\Omega \big|\nabla \mathbf{u}^\varepsilon(t)\big|^2 dx + \int_\Omega \rho^\varepsilon(x) \left| \frac{d\mathbf{u}^\varepsilon}{dt}(t) \right|^2 dx = \int_\Omega \big|\nabla \varphi^\varepsilon\big|^2 dx + \int_\Omega \rho^\varepsilon(x) |\psi|^2 \, dx, \quad \forall t > 0. \tag{4.6}$$

4.1. Background on the low frequencies

Convergence results for *the low frequencies*, the eigenvalues of order $O(\varepsilon^{m-2})$ of (4.2), and the associated eigenfunctions can be found in [32–34]. As in the case of one single concentrated mass, in general, the low frequencies are associated with *the local vibrations* of the concentrated masses, each one independent of the others. We have found only one exception: for $n = 3$ and $\alpha > 0$, these frequencies can also give rise to *global vibrations* affecting the whole structure (cf. [26] and [33]). Apart from this exception, the low frequencies and the corresponding eigenfunctions are asymptotically described, in a certain way, by the so called *local eigenvalue problem* (4.7).

The local problem is the spectral problem posed in $\mathbb{R}^{n-} = \{y \in \mathbb{R}^n \ / \ y_n < 0\}$:

$$\begin{cases} -\Delta_y U = \lambda U & \text{in } B, \\ -\Delta_y U = 0 & \text{in } \mathbb{R}^{n-} - \overline{B}, \\ [U] = \left[\dfrac{\partial U}{\partial n_y} \right] = 0 & \text{on } \Gamma, \\ U = 0 & \text{on } T, \quad \dfrac{\partial U}{\partial y_n} = 0 \quad \text{on } \{y_n = 0\} - \overline{T}, \\ U(y) \to c, & \text{as } |y| \to \infty, \quad y_n < 0 \quad \text{when } n = 2, \\ U(y) \to 0, & \text{as } |y| \to \infty, \quad y_n < 0 \quad \text{when } n = 3, \end{cases} \tag{4.7}$$

where the brackets denote the jump across Γ, $\bar{n}_y$ the unit outward normal to Γ and c some unknown but well determined constant. The variable y is the *local variable*:

$$y = \frac{x - \tilde{x}_k}{\varepsilon}. \tag{4.8}$$

Problem (4.7) can be written as a standard eigenvalue problem with a discrete spectrum in the space $\widetilde{\mathcal{V}}$, where $\widetilde{\mathcal{V}}$ is the completion of $\{U \in \mathcal{D}(\overline{\mathbb{R}^{n-}}) / U = 0 \text{ on } T\}$ for the Dirichlet norm $\|\nabla_y U\|_{L^2(\mathbb{R}^{n-})}$ (see [20] and [22]).

Theorem 4.1 below allows us to assert that there are at least $l_0 N(\varepsilon)$ values $\lambda^\varepsilon_{i(\varepsilon)}/\varepsilon^{m-2}$ converging towards each eigenvalue λ^0 of (4.7), l_0 being the multiplicity of λ^0. The corresponding eigenfunctions U^ε (cf. (4.9)) are approached in the space $\widetilde{\mathbf{V}}^\varepsilon$ by the eigenfunctions of (4.7) associated with λ^0, concentrating their support asymptotically in neighborhoods of the concentrated masses as stated in Theorem 4.1.

Also, the results in Theorem 2.2 along with results of comparison of the spectra for perturbed domains in [33] allow us to obtain an important difference for the asymptotic behavior of the low frequencies for the dimensions $n = 2$ and $n = 3$ of the space. Namely, for $n = 2$, and for each $i = 1, 2, \ldots, \lambda^\varepsilon_i/\varepsilon^{m-2} \to \lambda^0_1$, as $\varepsilon \to 0$, where λ^0_1 is the first eigenvalue of (4.7). This does not hold for $n = 3$ and $\alpha > 0$, α being the parameter defined in (4.1) (see [33] and [34] for further explanations). Let us refer to [12,16,30,29,38] to compare the above mentioned results with the stronger results on the approach for the eigenfunctions in the case of one single concentrated mass, case where the convergence of the re-scaled spectrum of (4.2) towards that of (4.7) with conservation of the multiplicity holds.

Let us change the variable in (4.2) by setting $y = x/\varepsilon$. We obtain:

$$\int_{\Omega_\varepsilon} \nabla_y U^\varepsilon . \nabla_y V^\varepsilon \, dy = \frac{\lambda^\varepsilon}{\varepsilon^{m-2}} \int_{\Omega_\varepsilon} \beta^\varepsilon(y) U^\varepsilon V^\varepsilon \, dy, \quad \forall V^\varepsilon \in \widetilde{\mathbf{V}}^\varepsilon, \tag{4.9}$$

Ω_ε being the domain $\{y / \varepsilon y \in \Omega\}$ and $\beta^\varepsilon(y)$ in (4.9) is defined as

$$\beta^\varepsilon(y) = 1 \quad \text{if } y \in \bigcup \tau_y B^\varepsilon, \quad \text{and} \quad \beta^\varepsilon(y) = \varepsilon^m \quad \text{if } y \in \Omega_\varepsilon - \overline{\bigcup \tau_y B^\varepsilon}, \tag{4.10}$$

where $\tau_y B^\varepsilon$ denote the transformed domains of the regions B^ε to the y variable. $\widetilde{\mathbf{V}}^\varepsilon$ is the functional space $\{U = U(y) / U(\varepsilon y) \in \mathbf{V}^\varepsilon\}$, with the norm defined by the right-hand side of (4.9). We assume that the eigenfunctions $\{U^\varepsilon_i\}_{i=1}^\infty$, in the local variable, satisfy $\|U_i{}^\varepsilon\|_{\widetilde{\mathbf{V}}^\varepsilon} = 1$.

Let us consider λ^0 an eigenvalue of (4.7) of multiplicity l_0 and let $U^0_1, U^0_2, \ldots, U^0_{l_0}$ be the corresponding eigenfunctions, orthogonal in $\widetilde{\mathcal{V}}$, satisfying $\|\nabla_y U^0_i\|_{L^2(\mathbb{R}^{n-})} = 1$.

Let us introduce $\tilde{\varphi}^\varepsilon(y)$ a function defined depending on the value of n. For $n = 2$, we consider $R_\varepsilon = \sqrt{(4\varepsilon + \eta)/4\varepsilon}$, and we define:

$$\tilde{\varphi}^\varepsilon(y) = \begin{cases} 1 & \text{if } |y| \leqslant R_\varepsilon, \\ 1 - \dfrac{\ln|y| - \ln R_\varepsilon}{\ln R_\varepsilon} & \text{if } R_\varepsilon \leqslant |y| \leqslant R^2_\varepsilon, \\ 0 & \text{if } |y| \geqslant R^2_\varepsilon. \end{cases} \tag{4.11}$$

For $n = 3$, we consider $\tilde{\varphi}^\varepsilon$ as a smooth function which takes the value 1 in the semi-ball of radius $((\varepsilon + \eta/8)/\varepsilon)$, $B((\varepsilon + \eta/8)/\varepsilon)$, and is zero outside the semi-ball of radius $((\varepsilon + \eta/4)/\varepsilon)$, $B((\varepsilon + \eta/4)/\varepsilon)$:

$$\tilde{\varphi}^\varepsilon(y) = \varphi\left(2 \frac{|\varepsilon y| - \varepsilon}{\eta}\right), \tag{4.12}$$

where $\varphi \in C^\infty[0,1]$, $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ in $[0, 1/4]$ and $\text{Supp}(\varphi) \subset [0, 1/2]$.

Obviously, the elements of $\widetilde{\mathbf{V}}^\varepsilon$ extended by zero in $\mathbb{R}^{n-} - \overline{\Omega_\varepsilon}$ are elements of $\widetilde{\mathcal{V}}$. Moreover, we have that $U^0_p \tilde{\varphi}^\varepsilon \in \widetilde{\mathbf{V}}^\varepsilon$, and, as $\varepsilon \to 0$, $U^0_p \tilde{\varphi}^\varepsilon \to U^0_p$ in $\widetilde{\mathcal{V}}$ (see [20,22]). For each $k = 1, 2, \ldots, N(\varepsilon)$, $p = 1, 2, \ldots, l_0$, we introduce the function (see [24,32] for this construction):

$$Z^\varepsilon_{k,p}(y) = \frac{U^0_p(y - \frac{\tilde{x}_k}{\varepsilon}) \tilde{\varphi}^\varepsilon(y - \frac{\tilde{x}_k}{\varepsilon})}{\|\nabla_y (U^0_p \tilde{\varphi}^\varepsilon)\|_{L^2(\mathbb{R}^{n-})}}, \tag{4.13}$$

and the order functions o_ε independent of k and p, o_ε tending to 0 as $\varepsilon \to 0$:

$$o_\varepsilon = C\left(\ln \frac{4\varepsilon + \eta}{4\varepsilon}\right)^{-\frac{1}{2}} \quad \text{when } n = 2, \tag{4.14}$$

$$o_\varepsilon = C \max\left\{\left(\frac{\varepsilon}{\eta}\right)^{\frac{1}{2}}, \varepsilon^{m-2}\right\} \quad \text{when } n = 3, \tag{4.15}$$

with the constant C independent of ε.

Theorem 4.1. *Let us consider* λ^0 *an eigenvalue of* (4.7) *of multiplicity* l_0*, and, let* $U_1^0, U_2^0, \ldots, U_{l_0}^0$ *be the corresponding eigenfunctions which are assumed to be orthonormal in* $\widetilde{\mathcal{V}}$*. For any* $K > 0$ *there is* $\varepsilon^*(K)$ *such that, for* $\varepsilon < \varepsilon^*(K)$*,* $K < l_0 N(\varepsilon)$ *and the interval* $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$ *contains eigenvalues of* (4.9)*,* $\lambda^\varepsilon_{i(\varepsilon)}/\varepsilon^{m-2}$*, with total multiplicity greater than or equal to* K*;* d^ε *is a certain sequence,* $d^\varepsilon \to 0$ *as* $\varepsilon \to 0$ *and the interval* $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$ *does not contain other eigenvalues of* (4.7) *different from* λ^0*. We denote by,*

$$\left\{\frac{\lambda^\varepsilon_{i(\varepsilon)+j}}{\varepsilon^{m-2}}\right\}_{j=1}^{I(\lambda^0,d^\varepsilon)} \quad \text{and} \quad \left\{U^\varepsilon_{i(\varepsilon)+j}\right\}_{j=1}^{I(\lambda^0,d^\varepsilon)}, \tag{4.16}$$

the sets of the eigenvalues of (4.7) *in* $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$ *and of the associated eigenfunctions.*

In addition, we have that for any β *such that* $0 < \beta < 1$*, and for* $d^\varepsilon = (o_\varepsilon)^\beta$*, there are* $l_0 N(\varepsilon)$ *functions,* $\{U^\varepsilon_{k,p}\}_{k=1,N(\varepsilon)}^{p=1,l_0}$*,* $U^\varepsilon_{k,p} \in \widetilde{\mathbf{V}}^\varepsilon$*, such that* $\|U^\varepsilon_{k,p}\|_{\widetilde{\mathbf{V}}^\varepsilon} = 1$*,* $U^\varepsilon_{k,p}$ *belongs to the eigenspace associated with all the eigenvalues in* $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$*, and*

$$\left\|U^\varepsilon_{k,p} - Z^\varepsilon_{k,p}\right\|_{\widetilde{\mathbf{V}}^\varepsilon} \leqslant 2(o_\varepsilon)^{1-\beta}. \tag{4.17}$$

In (4.17)*,* o_ε *is given by* (4.14) *when* $n = 2$ ((4.15) *when* $n = 3$)*,* $Z^\varepsilon_{k,p}$ *is defined by* (4.13) *and* $\tilde{\varphi}^\varepsilon(y)$ *is defined by* (4.11) *when* $n = 2$ ((4.12) *when* $n = 3$)*. These functions,* $\{U^\varepsilon_{k,p}\}_{k=1,N(\varepsilon)}^{p=1,l_0}$*, satisfy that for any extracted subset of* K *functions* $\{U^\varepsilon_{j_1}, U^\varepsilon_{j_2}, \ldots, U^\varepsilon_{j_K}\}$*, they are linearly independent functions.*

Theorem 4.1 has been proved in [32] (cf. also [24,34]) by applying Theorem 2.1 and results on *almost orthogonality* for the quasimodes. Note that formula (4.17) has been obtained from the fact that $(Z^\varepsilon_{k,p}, 1/\lambda^0)$ is a quasimode of remainder o_ε for the operator on $\widetilde{\mathbf{V}}^\varepsilon$ defined by the right-hand side of (4.9). In the same way, according to (2.2), the width of the interval $d^\varepsilon = (o_\varepsilon)^\beta$ and the bound (4.17) provide the closeness of these quasimodes and the eigenelements of the above mentioned operator.

Theorem 4.1 can be re-written in terms of the frequencies of the original problem (4.4) and the macroscopic variable x as follows:

Theorem 4.2. *Let us consider* λ^0 *an eigenvalue of* (4.7) *of multiplicity* l_0*, and, let* $U_1^0, U_2^0, \ldots, U_{l_0}^0$ *be the corresponding eigenfunctions which are assumed to be orthonormal in* $\widetilde{\mathcal{V}}$*. For any* $K > 0$ *there is* $\varepsilon^*(K)$ *such that, for* $\varepsilon < \varepsilon^*(K)$*,* $K < l_0 N(\varepsilon)$ *and the interval* $[\lambda^0\varepsilon^{m-2} - d^\varepsilon\varepsilon^{m-2}, \lambda^0\varepsilon^{m-2} + d^\varepsilon\varepsilon^{m-2}]$ *contains eigenvalues of* (4.4)*,* $\lambda^\varepsilon_{i(\varepsilon)}$*, with total multiplicity greater than or equal to* K*;* d^ε *is a certain sequence,* $d^\varepsilon \to 0$ *as* $\varepsilon \to 0$ *and the interval* $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$ *does not contain other eigenvalues of* (4.7) *different from* λ^0*.*

In addition, we have that for any β *such* $0 < \beta < 1$*, and for* $d^\varepsilon = (o_\varepsilon)^\beta$*, there are* $l_0 N(\varepsilon)$ *functions,* $\{u^\varepsilon_{k,p}\}_{k=1,N(\varepsilon)}^{p=1,l_0}$*,* $u^\varepsilon_{k,p} \in \mathbf{V}^\varepsilon$*, such that* $\|u^\varepsilon_{k,p}\|^2_{\mathbf{V}^\varepsilon} = \varepsilon^{n-2}$*,* $u^\varepsilon_{k,p}$ *belongs to the eigenspace associated with all the eigenvalues in* $[\lambda^0\varepsilon^{m-2} - d^\varepsilon\varepsilon^{m-2}, \lambda^0\varepsilon^{m-2} + d^\varepsilon\varepsilon^{m-2}]$*, and*

$$\left\|u^\varepsilon_{k,p} - \tau_x Z^\varepsilon_{k,p}\right\|_{H^1(\Omega)} \leqslant 2(o_\varepsilon)^{1-\beta}\varepsilon^{(n-2)/2}. \tag{4.18}$$

In (4.18)*,* o_ε *is given by* (4.14) *when* $n = 2$ ((4.15) *when* $n = 3$)*,* $\tau_x Z^\varepsilon_{k,p}$ *is defined by* (4.13) *with the change of variable* (4.8) *from* y *to* x*, and* $\tilde{\varphi}^\varepsilon(y)$ *is defined by* (4.11) *when* $n = 2$ ((4.12) *when* $n = 3$)*. The functions* $\{u^\varepsilon_{k,p}\}_{k=1,N(\varepsilon)}^{p=1,l_0}$*, are defined by* $u^\varepsilon_{k,p}(x) = U^\varepsilon_{k,p}(y)$ *which also amounts to* $u^\varepsilon_{k,p} = \tau_x U^\varepsilon_{k,p}$*, while* $\{U^\varepsilon_{k,p}\}_{k=1,N(\varepsilon)}^{p=1,l_0}$ *are the functions provided by Theorem* 4.1*, satisfying* (4.17)*. These functions,* $\{u^\varepsilon_{k,p}\}_{k=1,N(\varepsilon)}^{p=1,l_0}$*, satisfy that for any extracted subset of* K *functions* $\{u^\varepsilon_{j_1}, u^\varepsilon_{j_2}, \ldots, u^\varepsilon_{j_K}\}$*, they are linearly independent functions.*

4.2. The evolution problem and the low frequencies

Let us consider the set spaces $\widetilde{\mathbf{V}}^\varepsilon$ and $\widetilde{\mathbf{H}}^\varepsilon$, where $\widetilde{\mathbf{V}}^\varepsilon$ is defined in (4.9) and $\widetilde{\mathbf{H}}^\varepsilon = \{U(y)/U(\varepsilon y) \in L^2(\Omega)\}$ with the norm $\|(\beta^\varepsilon)^{1/2}u\|_{L^2(\varepsilon^{-1}\Omega)}$, β^ε being defined by (4.10). Let A^ε be the operator associated with the form on $\widetilde{\mathbf{V}}^\varepsilon$ arising on the left-hand side of (4.9). Let $(Z^\varepsilon_{k,p}, 1/\lambda^0)$ be the quasimodes constructed in Section 4.1, for $k = 1, 2, \ldots, N(\varepsilon)$, $p = 1, 2, \ldots, l_0$, from the eigenelement (λ^0, U^0_p) of the local problem (4.7).

Let us consider the second order evolution problem associated with (4.9):

$$\begin{cases} \dfrac{d^2\mathbf{U}^\varepsilon}{dt^2} + A^\varepsilon \mathbf{U}^\varepsilon = 0, \\ \mathbf{U}^\varepsilon(0) = \varphi^\varepsilon, \\ \dfrac{d\mathbf{U}^\varepsilon}{dt}(0) = \psi^\varepsilon. \end{cases} \tag{4.19}$$

For initial data $(\varphi^\varepsilon, \psi^\varepsilon) \in \widetilde{\mathbf{V}}^\varepsilon \times \widetilde{\mathbf{H}}^\varepsilon$, problem (4.19) has a unique solution $\mathbf{U}^\varepsilon(t)$ satisfying:

$$\mathbf{U}^\varepsilon \in L^\infty\big(0,\infty,\widetilde{\mathbf{V}}^\varepsilon\big), \qquad \frac{d\mathbf{U}^\varepsilon}{dt} \in L^\infty\big(0,\infty,\widetilde{\mathbf{H}}^\varepsilon\big), \qquad \mathbf{U}^\varepsilon(0) = \varphi^\varepsilon,$$

and, for any fixed $T > 0$,

$$\int_0^T \bigg(\int_{\varepsilon^{-1}\Omega} \nabla_y \mathbf{U}^\varepsilon . \nabla_y \boldsymbol{\Phi}\, dy - \int_{\varepsilon^{-1}\Omega} \beta^\varepsilon(y) \frac{d\mathbf{U}^\varepsilon}{dt}\frac{d\boldsymbol{\Phi}}{dt}\, dy \bigg) dt = \int_{\varepsilon^{-1}\Omega} \beta^\varepsilon(y)\psi^\varepsilon \boldsymbol{\Phi}(0)\, dy,$$

for any test function $\boldsymbol{\Phi}$ of the form $\boldsymbol{\Phi} = \phi(t)V$, where $V \in \widetilde{\mathbf{V}}^\varepsilon$, and $\phi \in C^1([0,T])/\phi(T) = 0$. In addition,

$$\big\|\mathbf{U}^\varepsilon(t)\big\|^2_{\widetilde{\mathbf{V}}^\varepsilon} + \bigg\|\frac{d\mathbf{U}^\varepsilon}{dt}(t)\bigg\|^2_{\widetilde{\mathbf{H}}^\varepsilon} = \|\varphi^\varepsilon\|^2_{\widetilde{\mathbf{V}}^\varepsilon} + \|\psi^\varepsilon\|^2_{\widetilde{\mathbf{H}}^\varepsilon}, \quad \forall t > 0 \tag{4.20}$$

see (2.4)–(2.8) for $(\mathbf{V},\mathbf{H}) \equiv (\widetilde{\mathbf{V}}^\varepsilon, \widetilde{\mathbf{H}}^\varepsilon)$ and $(\varphi,\psi) \equiv (\varphi^\varepsilon, \psi^\varepsilon)$).

We refer to [36] for the explicit construction of the operator $\mathcal{A}^\varepsilon$ for which the functions (4.13) arising in Theorem 4.1, namely $Z^\varepsilon_{k,p}$, are quasimodes. See [36] in connection with formulas (2.7)–(2.12), that is for the solutions of (4.19) which are standing waves or sums of standing waves when the initial data are linear combinations of eigenfunctions of (4.16) and for the Fourier expansion of the solutions of (4.19) for general data. In the case where the initial data are the quasimodes $Z^\varepsilon_{k,p}$ associated with the eigenelement (λ^0, U^0_p) of (4.7), for $k = 1,2,\dots,N(\varepsilon)$ and $p = 1,2,\dots,l_0$, approaching the functions $U^\varepsilon_{k,p}$ (cf. (4.17)), the solutions of the evolution problem (4.19) are not standing waves or sums of standing waves. The following theorem establishes the range of t where the standing wave $\cos(\sqrt{\lambda^0}t)Z^\varepsilon_{k,p}$ ($\sqrt{(\lambda^0)^{-1}}\sin(\sqrt{\lambda^0}t)Z^\varepsilon_{k,p}$, resp.) approaches the solution $\mathbf{U}^\varepsilon(t)$ of (4.19) for the initial data $(\varphi^\varepsilon, \psi^\varepsilon) = (Z^\varepsilon_{k,p}, 0)$ ($(\varphi^\varepsilon, \psi^\varepsilon) = (0, Z^\varepsilon_{k,p})$, resp.). See Remark 4.1 in this respect.

Theorem 4.3. *Let* (λ^0, U^0_p) *be an eigenelement of* (4.7)*, and* $Z^\varepsilon_{k,p}$ *defined by* (4.13) *for* $k = 1,2,\dots,N(\varepsilon)$*, and* $p = 1,2,\dots,l_0$. *Let us consider problem* (4.19) *for* $(\varphi^\varepsilon, \psi^\varepsilon) = (Z^\varepsilon_{k,p}, 0)$. *Then, for* $t > 0$*, and sufficiently small* ε (*namely,* $\varepsilon < \varepsilon_0$ *with* ε_0 *independent of* t)*, the unique solution* $\mathbf{U}^\varepsilon(t)$ *of* (4.19) *satisfies*:

$$\big\|\cos\big(\sqrt{\lambda^0}t\big)Z^\varepsilon_{k,p} - \mathbf{U}^\varepsilon(t)\big\|_{\widetilde{\mathbf{V}}^\varepsilon} \leqslant C_1 \max\big((o_\varepsilon)^{1-\beta}, (o_\varepsilon)^{\frac{\beta}{2}}t\big), \tag{4.21}$$

$$\bigg\|\sqrt{\lambda^0}\sin\big(\sqrt{\lambda^0}t\big)Z^\varepsilon_{k,p} + \frac{d\mathbf{U}^\varepsilon}{dt}(t)\bigg\|_{\widetilde{\mathbf{H}}^\varepsilon} \leqslant C_2 \max\big((o_\varepsilon)^{1-\beta}, (o_\varepsilon)^{\frac{\beta}{2}}t + (o_\varepsilon)^{\frac{\beta}{2}}\big), \tag{4.22}$$

where C_1 *and* C_2 *are constants that may depend on* λ^0*, but are independent of* ε *and* t*,* o_ε *is defined by* (4.14) *when* $n = 2$ *and by* (4.15) *when* $n = 3$*, and* β *is the constant appearing in* (4.17)*,* $0 < \beta < 1$.

In the same way, for $(\varphi^\varepsilon, \psi^\varepsilon) = (0, Z^\varepsilon_{k,p})$*, the following estimates hold*:

$$\bigg\|\frac{\sin(\sqrt{\lambda^0}t)}{\sqrt{\lambda^0}}Z^\varepsilon_{k,p} - \mathbf{U}^\varepsilon(t)\bigg\|_{\widetilde{\mathbf{V}}^\varepsilon} \leqslant C_1 \max\big((o_\varepsilon)^{1-\beta}, (o_\varepsilon)^{\frac{\beta}{2}}t + (o_\varepsilon)^{\frac{\beta}{2}}\big); \tag{4.23}$$

$$\bigg\|\cos\big(\sqrt{\lambda^0}t\big)Z^\varepsilon_{k,p} - \frac{d\mathbf{U}^\varepsilon}{dt}(t)\bigg\|_{\widetilde{\mathbf{H}}^\varepsilon} \leqslant C_2 \max\big((o_\varepsilon)^{1-\beta}, (o_\varepsilon)^{\frac{\beta}{2}}t\big). \tag{4.24}$$

Proof. The proof of Theorem 4.3 is now a consequence of Theorems 3.1 and 3.2 for the spaces $\mathcal{H}^\varepsilon = \widetilde{\mathbf{H}}^\varepsilon$ and $\mathcal{V}^\varepsilon = \widetilde{\mathbf{V}}^\varepsilon$, $r^\varepsilon = o_\varepsilon$, $r^*_\varepsilon = (o_\varepsilon)^\beta$ and the operator A^ε associated with the form on $\widetilde{\mathbf{V}}^\varepsilon$ arising on the left-hand side of (4.9). Indeed, first, let us note that (3.1) holds for C a constant independent of ε and u (see also Remark 3.1). In order to obtain (4.21) and (4.22), let us consider problem (4.19) for the initial data $(\varphi^\varepsilon, \psi^\varepsilon) = (Z^\varepsilon_{k,p} - U^\varepsilon_{k,p}, 0)$, where $U^\varepsilon_{k,p}$ are the functions in Theorem 4.1 satisfying (4.17), namely, $\varphi^\varepsilon = Z^\varepsilon_{k,p} - \sum_{j=1}^{I(\lambda^0,d^\varepsilon)} a^\varepsilon_{j,k,p} U^\varepsilon_{i(\varepsilon)+j}$, for certain constants $a^\varepsilon_{j,k,p}$. For these initial data, problem (4.19) has the solution:

$$\mathbf{U}^\varepsilon(t) - \sum_{j=1}^{I(\lambda^0,d^\varepsilon)} a^\varepsilon_{j,k,p} \cos\left(\sqrt{\frac{\lambda^\varepsilon_{i(\varepsilon)+j}}{\varepsilon^{m-2}}}\,t\right) U^\varepsilon_{i(\varepsilon)+j}.$$

Then, we use (3.10), (4.20), (3.3), (3.4), (3.8), and the precise bounds (4.17) (see also (4.11)–(4.17)), to obtain (4.21) and (4.22) from (3.11) and (3.12), respectively.

Similarly, we consider problem (4.19) for the initial data $(\varphi^\varepsilon, \psi^\varepsilon) = (0, Z^\varepsilon_{k,p} - U^\varepsilon_{k,p})$, which has the solution:

$$\mathbf{U}^\varepsilon(t) - \sum_{j=1}^{I(\lambda^0,d^\varepsilon)} a^\varepsilon_{j,k,p} \sqrt{\frac{\varepsilon^{m-2}}{\lambda^\varepsilon_{i(\varepsilon)+j}}} \sin\left(\sqrt{\frac{\lambda^\varepsilon_{i(\varepsilon)+j}}{\varepsilon^{m-2}}}\,t\right) U^\varepsilon_{i(\varepsilon)+j}.$$

From (3.5), (3.9) and (3.7) together with (4.20), (3.3), (3.4), (3.8) and (4.17) (see also (4.11)–(4.17)) we obtain (4.23) and (4.24). Therefore, the estimates in the theorem are proved. □

Remark 4.1. It should be noted that the bounds in Theorem 4.3, namely (4.21)–(4.24), establish the range of t where the standing waves $\cos(\sqrt{\lambda^0}t) Z^\varepsilon_{k,p}$ or $\sqrt{(\lambda^0)^{-1}}\sin(\sqrt{\lambda^0}t) Z^\varepsilon_{k,p}$ approach the solution of (4.19) for given initial data $(\varphi^\varepsilon, \psi^\varepsilon)$ certain quasimodes of the operator associated to problem (4.9). In fact, the approaches in Theorem 4.3 hold uniformly if

$$t \in \left[0, (o_\varepsilon)^{\frac{-\beta\beta'}{2}}\right],$$

for any constant β' satisfying $0 < \beta' < 1$. In this case, the bounds on the right-hand side of (4.21)–(4.24) are $C^*(o_\varepsilon)^{\min(1-\beta, \beta(1-\beta')/2)}$, C^* being a constant independent of ε.

In particular, both constants β and β' could be taken to be $\beta = \beta' = 1/2$, and if so, the bounds in the right-hand side of Theorem 4.3 are $C(o_\varepsilon)^{-1/8}$, for all $t < C^*(o_\varepsilon)^{1/8}$ and C and C^* constants independent of ε.

Remark 4.2. Note that all the bounds and results stated throughout the paper extend to the case where the concentrated masses are placed along a line or a plane inside the domain Ω. That is, B^ε has a smooth boundary which does not touch $\partial\Omega$, problem (4.2) reads:

$$\begin{cases} -\Delta u^\varepsilon = \rho^\varepsilon(x)\lambda^\varepsilon u^\varepsilon & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.25}$$

and (4.7) is the local problem, when $\mathbb{R}^{n-}$ is replaced by $\mathbb{R}^n$ (also Eq. $(4.7)_4$ disappears). Nevertheless, we also note that for (4.25) with $n = 2$ the low frequencies are of a lower order of magnitude than ε^{m-2}, as it has been shown in [34]. The same could happen for the dimension $n = 3$ (cf. [34] for the estimates and proofs of Theorem 4.1 for problem (4.25)).

Remark 4.3. We emphasize that the results in this paper are very different from those in [20] and [22], where evolution problems (4.19), for initial data $(\varphi^\varepsilon, \psi^\varepsilon) = (Z^\varepsilon_{1,1}, 0)$ are used to derive the spectral convergence of the low frequencies. Namely, using the Fourier transform, convergence results for the eigenelements of problem (4.2) are obtained. These convergence results prove to be much weaker than those stated in Theorem 4.1. In contrast, here, we use Theorem 4.1 to obtain approaches, via standing waves, to the solutions of (4.19) for given initial data which are the quasimodes constructed in Theorem 4.1. These approaches are valid for a large time which is established in Theorem 4.3 (see also Remark 4.1).

Remark 4.4. In the framework of Remark 4.2, it should be noted that in the case where we have a single concentrated mass inside Ω, or a finite number at a distance of order $O(1)$ between them, the convergence of the spectrum of (4.25) towards that of the associated local problem with conservation of the multiplicity has been proved in [16,30,29] and [38]. Also convergence rates for the eigenvalues and the associated eigenfunctions are outlined in [30] and [29]. That is, the r^ε, r^*_ε, d^ε are known and can be taken in such a way that there is only a fixed finite number of eigenvalues in the intervals associated with the quasimodes, which depend on the multiplicity of the eigenvalue of the local problem, independent of ε. Thus, the quasimodes in the initial data can be approaches to true eigenfunctions (individually).

The same proofs developed throughout the section show that we also have standing waves that approach solutions of the evolution problem for a long period of time. In this case, this period of time depends on the above mentioned discrepancies between the eigenvalues and eigenfunctions. This leads us to assert that in terms of the evolution problem, from a qualitative viewpoint, having approaches for the eigenfunctions u^ε through quasimodes amounts to having approaches of u^ε through eigenfunctions of the limiting problem.

4.2.1. On the low frequency vibrations in the macroscopic variable

We emphasize that computations throughout the section above and results in Theorem 4.3 involve re-scaled frequencies $\lambda^\varepsilon/\varepsilon^{m-2}$ of the original problem (4.2) and the local variable y in (4.8). This implies that the frequency of vibration of the standing waves constructed are of order $O(1)$. Therefore, considering the original problem (4.2) and the macroscopic variable, it is self-evident that frequencies for the associated waves must be very small, of the order $O(\varepsilon^{(m-2)/2})$. This is why it is reasonable to consider the evolution problem (4.5) associated with (4.2), with the initial data $(\tau_x Z^\varepsilon_{k,p}, 0)$ arising in Theorem 4.2, instead of (4.19).

Nevertheless, this causes problems when approaching the solutions of (4.5) if the initial velocities are provided by the quasimodes (namely, bounds related to (4.23)–(4.24) in the x variable). These problems are due to the fact that, in general, (3.1) does not hold for all the values of the parameter m with $m > 2$, and we cannot ensure that the quasimodes in the space $\widetilde{\mathbf{H}}^\varepsilon$ provide quasimodes in the space $\mathbf{H}^\varepsilon$ (see Theorems 4.1 and 4.2 to compare results in both spaces). As a consequence, we can only prove estimates in Theorem 4.4 below. That is, considering $(\tau_x Z^\varepsilon_{k,p}, 1/\lambda^0)$ the quasimodes constructed in Section 4.1, for $k = 1, 2, \ldots, N(\varepsilon)$, $p = 1, 2, \ldots, l_0$, from the eigenelement (λ^0, U^0_p) of the local problem (4.7), we take the initial data $(\varphi^\varepsilon, \psi^\varepsilon) = (\tau_x Z^\varepsilon_{k,p} 0)$ in (2.3) and obtain the bounds (4.26) and (4.27) below.

Theorem 4.4. *Let (λ^0, U^0_p) be an eigenelement of* (4.7)*, and $Z^\varepsilon_{k,p}$ defined by* (4.13) *for $k = 1, 2, \ldots, N(\varepsilon)$, and $p = 1, 2, \ldots, l_0$. Let us consider problem* (4.5) *for $(\varphi^\varepsilon \psi^\varepsilon) = (\tau_x Z^\varepsilon_{k,p}, 0)$. Then, for $t > 0$, and sufficiently small ε (namely, $\varepsilon < \varepsilon_0$ with ε_0 independent of t), the unique solution $\mathbf{u}^\varepsilon(t)$ of* (4.5) *satisfies*:

$$\left\|\cos\left(\sqrt{\lambda^0\varepsilon^{m-2}}t\right)\tau_x Z^\varepsilon_{k,p} - \mathbf{u}^\varepsilon(t)\right\|_{H^1(\Omega)} \leqslant C_1\varepsilon^{\frac{n-2}{2}} \max\left((o_\varepsilon)^{1-\beta}, (o_\varepsilon)^{\frac{\beta}{2}}\varepsilon^{\frac{m-2}{2}}t\right), \tag{4.26}$$

and

$$\begin{aligned}&\left\|\sqrt{\lambda^0\varepsilon^{m-2}}\sin\left(\sqrt{\lambda^0\varepsilon^{m-2}}t\right)\tau_x Z^\varepsilon_{k,p} + \frac{d\mathbf{u}^\varepsilon}{dt}(t)\right\|_{L^2(\Omega)}\\ &\quad\leqslant C_2\varepsilon^{\frac{n-2}{2}} \max\left((o_\varepsilon)^{1-\beta}, (o_\varepsilon)^{\frac{\beta}{2}}\varepsilon^{\frac{m-2}{2}}\left(\varepsilon^{\frac{m-2}{2}}t+1\right)\right),\end{aligned} \tag{4.27}$$

where C_1 and C_2 are constants that may depend on λ^0, but are independent of ε and t, o_ε is defined by (4.14) *when $n = 2$ and by* (4.15) *when $n = 3$, and β is the constant appearing in* (4.17)*, $0 < \beta < 1$.*

Proof. The theorem holds using the technique in Section 2, combining proofs of Theorems 2.2, 2.3, 3.1 and 3.2 and Corollaries 2.1 and 2.2, and taking into account (4.6) and (4.18). For brevity, we outline here the proof:

We consider $\psi^\varepsilon = 0$ and $\varphi^\varepsilon = \tau_x Z^\varepsilon_{k,p} - u^\varepsilon_{k,p}$, where $u^\varepsilon_{k,p} = \tau_x U^\varepsilon_{k,p}$ and $\tau_x Z^\varepsilon_{k,p}$ are the functions arising in Theorem 4.2, as in the proof of Theorem 4.3, $\varphi^\varepsilon = \tau_x Z^\varepsilon_{k,p} - \sum_{j=1}^{I(\lambda^0,d^\varepsilon)} a^\varepsilon_{j,k,p} u^\varepsilon_{i(\varepsilon)+j}$ for certain constants $a^\varepsilon_{j,k,p}$. For these initial data, problem (4.5) has the solution:

$$\mathbf{u}^\varepsilon(t) - \sum_{j=1}^{I(\lambda^0,d^\varepsilon)} a^\varepsilon_{j,k,p}\cos\left(\sqrt{\lambda^\varepsilon_{i(\varepsilon)+j}}t\right)u^\varepsilon_{i(\varepsilon)+j}.$$

Therefore, for any fixed $t > 0$, we can write:

$$\begin{aligned}
&\left\|\cos\left(\sqrt{\lambda^0\varepsilon^{m-2}}t\right)\nabla_x\left(\tau_x Z^\varepsilon_{k,p}\right) - \nabla_x \mathbf{u}^\varepsilon(t)\right\|_{L^2(\Omega)} \\
&\quad\leqslant \left\|\cos\left(\sqrt{\lambda^0\varepsilon^{m-2}}t\right)\nabla_x(\tau_x Z^\varepsilon_{k,p}) - \sum_{j=1}^{I(\lambda^0,d^\varepsilon)} a^\varepsilon_{j,k,p}\cos\left(\sqrt{\lambda^\varepsilon_{i(\varepsilon)+j}}t\right)\nabla_x u^\varepsilon_{i(\varepsilon)+j}\right\|_{L^2(\Omega)} \\
&\qquad + \left\|\sum_{j=1}^{I(\lambda^0,d^\varepsilon)} a^\varepsilon_{j,k,p}\cos\left(\sqrt{\lambda^\varepsilon_{i(\varepsilon)+j}}t\right)\nabla_x u^\varepsilon_{i(\varepsilon)+j} - \nabla_x \mathbf{u}^\varepsilon(t)\right\|_{L^2(\Omega)}. \qquad (4.28)
\end{aligned}$$

Using (4.6) and (4.18), the last term on the right-hand side of (4.28) is bounded by a constant times $(o_\varepsilon)^{1-\beta}\varepsilon^{(n-2)/2}$, while for the first one we can write:

$$\begin{aligned}
&\left\|\cos\left(\sqrt{\lambda^0\varepsilon^{m-2}}t\right)\nabla_x\left(\tau_x Z^\varepsilon_{k,p}\right) - \sum_{j=1}^{I(\lambda^0,d^\varepsilon)} a^\varepsilon_{j,k,p}\cos\left(\sqrt{\lambda^\varepsilon_{i(\varepsilon)+j}}t\right)\nabla_x u^\varepsilon_{i(\varepsilon)+j}\right\|_{L^2(\Omega)} \\
&\quad\leqslant \left\|\cos\left(\sqrt{\lambda^0\varepsilon^{m-2}}t\right)\nabla_x\left(\tau_x Z^\varepsilon_{k,p}\right) - \sum_{j=1}^{I(\lambda^0,d^\varepsilon)} a^\varepsilon_{j,k,p}\cos\left(\sqrt{\lambda^0\varepsilon^{m-2}}t\right)\nabla_x u^\varepsilon_{i(\varepsilon)+j}\right\|_{L^2(\Omega)} \\
&\qquad + \left\|\sum_{j=1}^{I(\lambda^0,d^\varepsilon)} a^\varepsilon_{j,k,p}\cos\left(\sqrt{\lambda^0\varepsilon^{m-2}}t\right)\nabla_x u^\varepsilon_{i(\varepsilon)+j} - \sum_{j=1}^{I(\lambda^0,d^\varepsilon)} a^\varepsilon_{j,k,p}\cos\left(\sqrt{\lambda^\varepsilon_{i(\varepsilon)+j}}t\right)\nabla_x u^\varepsilon_{i(\varepsilon)+j}\right\|_{L^2(\Omega)}.
\end{aligned}$$

We use again (4.18) for the first term in the above inequality, and we apply the Taylor series error of the cosinus function in a neighborhood of $\sqrt{\lambda^0\varepsilon^{m-2}}t$ for the second term, while using the fact that the set of eigenvalues $\lambda^\varepsilon_{i(\varepsilon)+j}$, for $j = 1, 2, \dots, I(\lambda^0, d^\varepsilon)$, belong to the interval $[\lambda^0\varepsilon^{m-2} - d^\varepsilon\varepsilon^{m-2}, \lambda^0\varepsilon^{m-2} + d^\varepsilon\varepsilon^{m-2}]$; then, we obtain:

$$\begin{aligned}
&\left\|\cos\left(\sqrt{\lambda^0\varepsilon^{m-2}}t\right)\nabla_x\left(\tau_x Z^\varepsilon_{k,p}\right) - \nabla_x \mathbf{u}^\varepsilon(t)\right\|_{L^2(\Omega)} \\
&\quad\leqslant c_1(o_\varepsilon)^{1-\beta}\varepsilon^{(n-2)/2} + \varepsilon^{(n-2)/2}\max_{1\leqslant j\leqslant I(\lambda^0,d^\varepsilon)}\left|\sqrt{\lambda^\varepsilon_{i(\varepsilon)+j}}t - \sqrt{\lambda^0\varepsilon^{m-2}}t\right| \\
&\quad\leqslant c_1(o_\varepsilon)^{1-\beta}\varepsilon^{(n-2)/2} + c_2\varepsilon^{(n-2)/2}(o_\varepsilon)^{\beta/2}\varepsilon^{(m-2)/2},
\end{aligned}$$

where c_1 and c_2 are two constants independent of ε and t. Therefore the inequality (4.26) also holds on account of the Poincaré inequality for the elements of $\mathbf{V}^\varepsilon$.

As regards (4.27), we take into account,

$$\|u\|_{L^2(\Omega)} < \|u\|_{\mathbf{H}^\varepsilon}, \quad \forall u \in L^2(\Omega),$$

and the Poincaré inequality for the elements of $\mathbf{V}^\varepsilon$. Then, we follow the steps for the proofs of (2.14), (2.23) and (4.26) on account of (4.6), (4.18) and we have:

$$\begin{aligned}
&\left\|\sqrt{\lambda^0\varepsilon^{m-2}}\sin\left(\sqrt{\lambda^0\varepsilon^{m-2}}t\right)\tau_x Z^\varepsilon_{k,p} + \frac{d\mathbf{u}^\varepsilon}{dt}(t)\right\|_{L^2(\Omega)} \\
&\quad\leqslant c_3\varepsilon^{\frac{n-2}{2}}(o_\varepsilon)^{1-\beta} + c_4\sqrt{\lambda^0\varepsilon^{m-2}}\varepsilon^{\frac{n-2}{2}}(o_\varepsilon)^{1-\beta} \\
&\qquad + c_5\varepsilon^{\frac{n-2}{2}}\max_{1\leqslant j\leqslant I(\lambda^0,d^\varepsilon)}\left|\sqrt{\lambda^\varepsilon_{i(\varepsilon)+j}}\sin\sqrt{\lambda^\varepsilon_{i(\varepsilon)+j}}t - \sqrt{\lambda^0\varepsilon^{m-2}}\sin\sqrt{\lambda^0\varepsilon^{m-2}}t\right|,
\end{aligned}$$

where c_3, c_4 and c_5 are constants independent of t and ε. Now, we consider the term accompanying c_5 and write:

$$\begin{aligned}
&\max_{1\leqslant j\leqslant I(\lambda^0,d^\varepsilon)}\left|\sqrt{\lambda^\varepsilon_{i(\varepsilon)+j}}\sin\sqrt{\lambda^\varepsilon_{i(\varepsilon)+j}}t - \sqrt{\lambda^0\varepsilon^{m-2}}\sin\sqrt{\lambda^0\varepsilon^{m-2}}t\right| \\
&\quad\leqslant \max_{1\leqslant j\leqslant I(\lambda^0,d^\varepsilon)}\left|\sqrt{\lambda^\varepsilon_{i(\varepsilon)+j}}\sin\sqrt{\lambda^\varepsilon_{i(\varepsilon)+j}}t - \sqrt{\lambda^0\varepsilon^{m-2}}\sin\sqrt{\lambda^\varepsilon_{i(\varepsilon)+j}}t\right| \\
&\qquad + \max_{1\leqslant j\leqslant I(\lambda^0,d^\varepsilon)}\left|\sqrt{\lambda^0\varepsilon^{m-2}}\sin\sqrt{\lambda^\varepsilon_{i(\varepsilon)+j}}t - \sqrt{\lambda^0\varepsilon^{m-2}}\sin\sqrt{\lambda^0\varepsilon^{m-2}}t\right|,
\end{aligned}$$

and, then, the Taylor series error of the sinus function in a neighborhood of $\sqrt{\lambda^0\varepsilon^{m-2}}t$, and the fact that the set of eigenvalues $\lambda^\varepsilon_{i(\varepsilon)+j}$, for $j=1,2,\ldots,I(\lambda^0,d^\varepsilon)$ belong to the interval $[\lambda^0\varepsilon^{m-2}-d^\varepsilon\varepsilon^{m-2},\lambda^0\varepsilon^{m-2}+d^\varepsilon\varepsilon^{m-2}]$, lead us to obtain (4.27). Thus, the theorem is proved. □

Remark 4.5. According to Remark 4.1, estimates (4.26)–(4.27), establish the range of t where the standing waves of the type $\cos(\sqrt{\lambda^0\varepsilon^{m-2}}t)\tau_x Z^\varepsilon_{k,p}$ approach the solution of (4.5) for the given initial data in Theorem 4.4. In fact, the approaches in Theorem 4.3 hold uniformly if,

$$t\in\left[0,(o_\varepsilon)^{\frac{-(\beta+n+m-4)\beta'}{2}}\right]$$

for any constant β' satisfying $0<\beta'<1$. In this case, the bounds on the right-hand side of (4.26)–(4.27) are $C^*(o_\varepsilon)^{\min(1-\beta+(n-2)/2,(1-\beta')(\beta+n+m-4)/2)}$, C^* being a constant independent of ε.

4.3. Background and approaches to high frequency vibrations

In this section we deal with the eigenvalues of order $O(1)$ of (4.2), namely, $\lambda^\varepsilon=\lambda^\varepsilon_{i(\varepsilon)}=O(1)$. We assume that the associated eigenfunctions $u^\varepsilon=u^\varepsilon_{i(\varepsilon)}$ have norm 1 in $\mathbf{V}^\varepsilon$, which amounts to being bounded in $\mathbf{H}^\varepsilon$ (see (4.4)). In Theorem 4.5 we gather the main results of use in Theorem 4.6 to construct standing waves which approach the solutions of the evolution problem (4.5), for certain initial data and for long times. We refer to [26] and to [25] for a review and a large list of results dealing with the converging sequences of the eigenelements $(\lambda^\varepsilon,u^\varepsilon)$ as $\varepsilon\to0$, when $\lambda^\varepsilon=O(1)$.

As a matter of fact, a homogenized problem arises related with the high frequency vibrations. The homogenized problem depends on the dimension of the space n and on the relation (4.1) between the parameters ε and η. For the *critical size* of the masses B^ε, namely for $\alpha>0$, the homogenized problem is:

$$\begin{cases}-\Delta u=\lambda u & \text{in } \Omega,\\ u=0 & \text{on } \Gamma_\Omega,\\ \dfrac{\partial u}{\partial n}=-\alpha\mathcal{C}u & \text{on } \Sigma,\end{cases}\tag{4.29}$$

where the constant $\mathcal{C}$ takes the value $\mathcal{C}=S_n/2$ with S_n the surface of the unit sphere in $\mathbb{R}^n$: $\mathcal{C}=\pi$ when $n=2$, and $\mathcal{C}=2\pi$ when $n=3$ (see Remark 4.7 for different geometries of B^ε).

For the extreme cases, the boundary condition on Σ in (4.29) reads:

$$\frac{\partial u}{\partial n}=0\quad\text{on } \Sigma,\ \text{when } \alpha=0,\tag{4.30}$$

$$u=0\quad\text{on } \Sigma,\ \text{when } \alpha=+\infty.\tag{4.31}$$

As is well known, all tree problems have a pure point spectrum which we denote by $\{\lambda_i\}_{i=1}^\infty$. We refer to [25] for an extensive literature on the relation of these three problems with the high frequency vibrations, further references, validity of these limit problems for different values of m (here we consider $m>2$), other geometries of the domains Ω and B^ε, and also for the proof of Theorem 4.5. For brevity, we state this theorem (also Theorem 4.6) for $\alpha>0$ but the main results also hold for $\alpha=0$ and $\alpha=\infty$, for other geometries of B^ε and for the extreme cases (see Remarks 4.7 and 4.8)

For $n=2$ we consider w^ε the function defined in $B(0,\frac{\eta}{2})$:

$$w^\varepsilon(x)=\begin{cases}0, & \text{if } |x|\leqslant\varepsilon,\\ \dfrac{\ln|x|-\ln\varepsilon}{\ln(\eta/2)-\ln\varepsilon}, & \text{if } \varepsilon\leqslant|x|\leqslant\frac{\eta}{2},\\ 1, & \text{if } |x|\geqslant\frac{\eta}{2}.\end{cases}\tag{4.32}$$

For $n=3$ we consider w^ε the function defined in $B(0,\frac{\eta}{2})$:

$$w^\varepsilon(x)=\begin{cases}0, & \text{if } |x|\leqslant\varepsilon,\\ \dfrac{|x|^{-1}-\varepsilon^{-1}}{2\eta^{-1}-\varepsilon^{-1}}, & \text{if } \varepsilon\leqslant|x|\leqslant\frac{\eta}{2},\\ 1, & \text{if } |x|\geqslant\frac{\eta}{2}.\end{cases}\tag{4.33}$$

We extend w^ε by periodicity to all the balls centered at the points $\tilde{x}_k$ of radius $\eta/2$ and by 1 outside these balls.

Theorem 4.5. *The following results hold*:

- *For each $\lambda^* > 0$, there is a sequence $\lambda^\varepsilon_{i(\varepsilon)}$ of eigenvalues of* (4.4) *converging towards λ^* as $\varepsilon \to 0$.*
- *Let λ be any positive real number. Let $I(\delta^\varepsilon)$ denote the interval $[\lambda - \delta^\varepsilon, \lambda + \delta^\varepsilon]$ having eigenvalues λ^ε of* (4.4), *and δ^ε converging to 0 as $\varepsilon \to 0$. Then, λ is an eigenvalue of the homogenized problem* (4.29) *if and only if there are $\{\delta^\varepsilon\}_\varepsilon$ and $\{\tilde{u}^\varepsilon\}_\varepsilon$, each $\tilde{u}^\varepsilon$ belonging to the eigenspace associated with all the eigenvalues λ^ε in $I(\delta^\varepsilon)$, $\tilde{u}^\varepsilon$ of norm 1 in $\mathbf{V}^\varepsilon$, and, such that $\|\tilde{u}^\varepsilon\|_{L^2(\Omega)} > a > 0$ for some constant a independent of ε.*
- *Let $\lambda > 0$ be an eigenvalue of the homogenized problem* (4.29), *and u^0 be an associated eigenfunction satisfying $\|u^0\|_{L^2(\Omega)} = 1$. Then, there is a sequence δ^ε, $\delta^\varepsilon \to 0$ as $\varepsilon \to 0$, such that the interval $[\lambda - \delta^\varepsilon, \lambda + \delta^\varepsilon]$ has eigenvalues of* (4.4). *Moreover, δ^ε can be chosen such that there is $\tilde{u}^\varepsilon$,*
$$\|\tilde{u}^\varepsilon\|_{\mathbf{H}^\varepsilon} = 1 \quad \text{and} \quad \|\tilde{u}^\varepsilon\|_{\mathbf{V}^\varepsilon} \leqslant C^*,$$
with C^* a constant independent of ε, $\tilde{u}^\varepsilon$ belonging to the eigenspace associated with all the eigenvalues λ^ε in $[\lambda - \delta^\varepsilon, \lambda + \delta^\varepsilon]$ such that
$$\tilde{u}^\varepsilon \xrightarrow{\varepsilon \to 0} u^0 \quad \text{in } H^1(\Omega)\text{-weak}. \tag{4.34}$$
- *Let us consider the situation above. Namely, let $\lambda > 0$ be an eigenvalue of the homogenized problem* (4.29), *and u^0 the corresponding eigenfunction with $\|u^0\|_{L^2(\Omega)} = 1$. Let us assume that the domain Ω is such that $u^0 \in C(\overline{\Omega})$. Let us consider w^ε the sequence of functions defined by* (4.32) *when $n = 2$ and* (4.33) *when $n = 3$. Then, the convergence of $\tilde{u}^\varepsilon$ towards u^0 stated above reads*:
$$\|\tilde{u}^\varepsilon - u^0 w^\varepsilon\|_{\mathbf{H}^\varepsilon} + \|\tilde{u}^\varepsilon - u^0 w^\varepsilon\|_{\mathbf{V}^\varepsilon} = \mathbf{o}_\varepsilon \xrightarrow{\varepsilon \to 0} 0, \tag{4.35}$$
and, consequently, also
$$\|\tilde{u}^\varepsilon - u^0 w^\varepsilon\|_{H^1(\Omega)} \xrightarrow{\varepsilon \to 0} 0. \tag{4.36}$$

As a matter of fact, the two last assertions in Theorem 4.5 are obtained from the fact that given an eigenelement (λ, u^0) of (4.29), then (λ^0, u^0) $((\lambda^0, u^0 w^\varepsilon$, respectively) is a quasimode for problem (4.2). Namely, a quasimode for a certain operator associated with (4.4) in the space $L^2(\Omega)$ ($\mathbf{V}^\varepsilon$, $\mathbf{H}^\varepsilon$ or $H^1(\Omega)$, respectively) with a certain reminder r^ε that cannot be computed explicitly in terms of known order functions of ε. The same can be said for $\mathbf{o}_\varepsilon$ and δ^ε. In fact, the results in [25] provide a certain relation between $\mathbf{o}_\varepsilon$ and δ^ε: δ^ε can be any sequence converging towards zero such that $\tilde{\mathbf{o}}_\varepsilon < \delta^\varepsilon$, for a certain well determined $\tilde{\mathbf{o}}_\varepsilon$, and $\mathbf{o}_\varepsilon = \tilde{\mathbf{o}}_\varepsilon/\delta^\varepsilon$ (we can take $\mathbf{o}_\varepsilon = \delta^\varepsilon = \sqrt{\tilde{\mathbf{o}}_\varepsilon}$). Also, we emphasize that the condition $u^0 \in C(\overline{\Omega})$ is not a very restrictive condition (cf. Lemma 4 in [25]).

Theorem 4.6. *Let us consider (λ, u^0) an eigenelement of* (4.29), *with $\|u^0\|_{L^2(\Omega)} = 1$, and w^ε the functions defined by* (4.32) *when $n = 2$ and* (4.33) *when $n = 3$.*

Considering (4.5) *for the initial data $(\varphi^\varepsilon, \psi^\varepsilon) = (u^0 w^\varepsilon, 0)$, for any $t > 0$, and sufficiently small ε, namely $\varepsilon < \varepsilon_0$ with ε_0 independent of t, the solution $\mathbf{u}^\varepsilon(t)$ of* (4.5) *satisfies*:
$$\|\cos(\sqrt{\lambda} t) u^0 w^\varepsilon - \mathbf{u}^\varepsilon(t)\|_{H^1(\Omega)} \leqslant C_1 \max(\mathbf{o}_\varepsilon, \sqrt{\delta^\varepsilon} t), \tag{4.37}$$
and
$$\left\| \sqrt{\rho^\varepsilon(x)} \left(\sqrt{\lambda} \sin(\sqrt{\lambda} t) u^0 w^\varepsilon + \frac{d\mathbf{u}^\varepsilon}{dt}(t) \right) \right\|_{L^2(\Omega)} \leqslant C_2 \max(\mathbf{o}_\varepsilon, \sqrt{\delta^\varepsilon}(t+1)). \tag{4.38}$$

Considering (4.5) *for the initial data $(\varphi^\varepsilon, \psi^\varepsilon) = (0, u^0 w^\varepsilon)$, for any $t > 0$, and sufficiently small ε, namely $\varepsilon < \varepsilon_0$ with ε_0 independent of t, the solution $\mathbf{u}^\varepsilon(t)$ of* (4.5) *satisfies*:
$$\left\| \sqrt{\rho^\varepsilon(x)} \left(\frac{\sin(\sqrt{\lambda} t)}{\sqrt{\lambda}} u^0 w^\varepsilon - \mathbf{u}^\varepsilon(t) \right) \right\|_{L^2(\Omega)} \leqslant C_3 \max(\mathbf{o}_\varepsilon, \sqrt{\delta^\varepsilon}(t+1)), \tag{4.39}$$
$$\left\| \frac{\sin(\sqrt{\lambda} t)}{\sqrt{\lambda}} u^0 w^\varepsilon - \mathbf{u}^\varepsilon(t) \right\|_{H^1(\Omega)} \leqslant C_4 \max(\mathbf{o}_\varepsilon, \sqrt{\delta^\varepsilon}(t+1)), \tag{4.40}$$

and

$$\left\| \sqrt{\rho^\varepsilon(x)} \left(\cos(\sqrt{\lambda} t) u^0 w^\varepsilon - \frac{d\mathbf{u}^\varepsilon}{dt}(t) \right) \right\|_{L^2(\Omega)} \leqslant C_5 \max\left(\mathbf{o}_\varepsilon, \sqrt{\delta^\varepsilon} t\right). \tag{4.41}$$

Here, C_1, C_2, C_3, C_4 and C_5 are constants that may depend on λ but which are independent of t and ε. $\mathbf{o}_\varepsilon$ and δ^ε are the sequences arising in the last statement of Theorem 4.5.

Proof. Taking into account (4.6), the theorem is a direct consequence of (4.35) in Theorem 4.5 and of the application of Theorems 3.1 and 3.2 for the spaces $\mathbf{V}^\varepsilon$ and $\mathbf{H}^\varepsilon$, which are defined in (4.3)–(4.6), and for the values of r^ε and r_ε^*: $r^\varepsilon / r_\varepsilon^* = \mathbf{o}_\varepsilon$ and $r_\varepsilon^* = \delta^\varepsilon$.

Indeed, taking into account that the norm in $\mathbf{V}^\varepsilon$ is that of the gradient and that the Poincaré inequality holds for the elements of $\mathbf{V}^\varepsilon$, we use (3.11), (3.12) to derive (4.37) and (4.38) and (3.6), (3.7) and (3.9) to derive (4.39)–(4.41). Therefore, all the estimates in the theorem hold. □

Remark 4.6. Bounds (4.37)–(4.41) lead us to assert that for the initial data in Theorem 4.6 the solutions of (4.5) asymptotically remain at rest inside the concentrated masses for times $t \in [0, (\delta^\varepsilon)^{-\beta'}]$, with any β', $0 < \beta' < 1/2$.

Remark 4.7. Let us denote by $\widetilde{\Omega}$ ($\widetilde{B}^\varepsilon$, $\widetilde{\Gamma}^\varepsilon$, respectively) the extended domain of Ω (B^ε, Γ^ε, respectively) by symmetry to $\mathbb{R}^n$. In the case where the B^ε are not half-balls, that is, B is any open bounded domain of $\mathbb{R}^n$ with a Lipschitz boundary, T the part in contact with $\{x_n = 0\}$ and Γ the rest, the test functions w^ε must be replaced by the functions satisfying:

$$\begin{cases} -\Delta w^\varepsilon = 0 & \text{in } B(0, \frac{\eta}{2}) \setminus \widetilde{B}^\varepsilon, \\ w^\varepsilon = 0 & \text{in } \widetilde{B}^\varepsilon, \\ w^\varepsilon = 1 & \text{on } \partial B(0, \frac{\eta}{2}), \end{cases} \tag{4.42}$$

and the constant $\mathcal{C}$ appearing in (4.29) must be replaced by the *capacity constant*,

$$\mathcal{C} = \int_{\mathbb{R}^{3-} \setminus B} |\nabla_y U|^2 \, dy,$$

when $n = 3$, where U is the solution of a stationary local problem, U satisfying Eqs. $(4.7)_2$, $(4.7)_4$, $(4.7)_6$ and the equation in B: $U \equiv 1$ (consequently, also $U = 1$ on T). The constant $\mathcal{C}$ remains $\mathcal{C} = \pi$ when $n = 2$. We refer to [25], and more specifically to Remark 7 in [25], for comparison of proofs and explicit formulas that must be combined with the maximum principle and compactness results for non-negative measures (see also [3,10,28] for the technique).

Remark 4.8. Results in Theorem 4.5 hold for $\alpha = 0$ and $\alpha = +\infty$ with the following modifications: Condition on Σ in (4.29) is replaced by (4.30) and (4.31), respectively, (4.34) holds in the strong topology of $H^1(\Omega)$, and, in the case where $\alpha = +\infty$, convergence (4.35) and (4.36) reads:

$$\left\| \tilde{u}^\varepsilon - u^0 w^\varepsilon \right\|_{\mathbf{H}^\varepsilon} \xrightarrow{\varepsilon \to 0} 0. \tag{4.43}$$

Consequently, the results of Theorem 4.6 hold for $\alpha = 0$, while for $\alpha = +\infty$ the results have to be derived from (3.6) and (3.7) for $u^\varepsilon \equiv u^0 w^\varepsilon$ and from (3.15) for $u^\varepsilon \equiv u^0$.

Remark 4.9. Roughly speaking, Theorem 4.5 shows that the high frequencies accumulate on the whole positive real axis $(0, \infty)$ and singles out the eigenvalues of the homogenized problems from the others depending on the behavior of the associated eigenfunctions, since it shows that only the eigenfunctions u^ε associated with the eigenvalues λ^ε asymptotically near an eigenvalue of the homogenized problem are asymptotically different from zero in $L^2(\Omega)$.

In this respect, it should be emphasized that, from a qualitative viewpoint, these results are in fact very general results for many problems of spectral perturbation theory such as stiff problems or problems with one single concentrated mass (cf. [11,12] and [23]). In addition, these results involve the energy norm.

Remark 4.10. Also, in connection with Remark 4.9, we note that the asymptotic spectral accumulation of the eigenvalues λ^ε on the whole real positive axis could recall the so-called *spectral pollution*. That is when a sequence of eigenvalues of a perturbed operator approach a point of the resolvent of the original operator: let us refer to [37] for the precise definition of spectral pollution, connections with the Weyl sequences and convergences of the spectral families as well as for further references on the subject. See also Section III.2 in [1] for related issues.

Nevertheless, the results in Theorem 4.5 are not in the scope of the spectral pollution since our original problem is already the perturbed problem and we try to approach its eigenvalues. These eigenvalues can in fact be approached through the points of the continuous spectrum of the Laplacian in an unbounded domain (cf. [12] and [25]), the eigenvalues $\{\lambda_i\}_{i=1}^\infty$ of the homogenized problem (namely, (4.30)–(4.31)) being very particular points in the way stated in Theorems 4.5 and 4.6.

For the rest of the values $\lambda \notin \{\lambda_i\}_{i=1}^\infty$, depending on the geometrical configuration of the problem, it is possible to construct explicitly functions ψ^ε such that $\|\tilde{u}^\varepsilon - \psi^\varepsilon\|_{H^1(\Omega)} \to 0$ as $\varepsilon \to 0$, for a certain sequence of δ^ε in the second statement of Theorem 4.5 and ε ranging in certain subsequences still denoted by ε (see Remark 4.11 and [34] for further references).

Remark 4.11. Theorem 4.5 also holds in the case where we have one single concentrated mass, either near the boundary or inside Ω (cf. (4.25)). If so the homogenized problem is the classical spectral Dirichlet problem in Ω (namely, as if $\alpha = 0$), and o_ε and δ^ε can be explicitly computed from the results for the high frequencies in [12].

For problem (4.25), other correcting terms, improving the convergence (4.34), (4.35) and (4.36) (also the convergence of $\tilde{u}^\varepsilon$ towards zero in $H^1(\Omega)$-weak, in the case where λ is not an eigenvalue of the Dirichlet problem in Ω) can be constructed. We refer to [12] and [31] for the explicit computation of these correcting terms in the case where $n = 2$. Here, we only mention, that these correcting terms restrict the approaches to certain subsequences of ε and involve eigenfunctions of the corresponding local problem associated with very large eigenvalues (cf. Remark 4.2).

In terms of the evolution problem (cf. Theorem 4.6), when the initial data are certain specific quasimodes, it seems as if, for certain ε, the high frequency vibrations of (4.25) would interact with the high frequency vibrations of the local problem, and this allows us to construct standing waves for long times having a very short wavelength inside the concentrated mass, or concentrating exclusively their support in a thin layer along the interface Γ^ε, that is, along the boundary of the concentrated mass.

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